Two-Player Zero Sum Games Played on Graphs: $\omega$-Regular and Quantitative Objectives

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Controller synthesis as solving a game

- support the design process with **automatic synthesis**

Sys is constructed by an **algorithm**
Sys is **correct** by construction
Underlying theory: **2-player zero-sum games**
Env is **adversarial** (worst-case assumption)

Winning strategy = Correct Sys
Controller versus Env

Vertices model states of the system (=controller + environment)

Edges controlled by controller = choices for actions of the controller

Edges controlled by the environment = choices for actions of the environment

Winning objective for Controller (Player 1) = specification that the controller must enforce no matter what is the behavior of the environment (worst-case assumption)

$$\text{Win}_1 = G(S_E \rightarrow x_E \oplus y_C)$$
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Part II: Quantitative Games
Weighted game arena

Directed graph with $\mathbb{Z}$-weighted edges $A=(S_1,S_2,s_{init},E,w)$

As for qualitative games, the game is played in rounds:
- initially a token is on state $s_{init}$,
- rounds: the player owning the current state chooses an outgoing edge to move the token

The outcome is an infinite path together with an infinite sequence of $\mathbb{Z}$-weights
Winning condition: the set of infinite paths is partitioned into
- $W_1=$ winning for player 1
- $W_2=S_\omega \setminus W_1=$ winning for player 2

defined using a quantitative measure and a threshold

: states of Player 1 - maximizer=controller

: states of Player 2 - minimizer=environment
Quantitative measures

• in a weighted game arena, each infinite play $\rho = s_0 e_0 s_1 e_1 \ldots s_n e_n s_{n+1} \ldots$ defines an infinite sequence of weights $w(e_0) w(e_1) \ldots w(e_n) \ldots \in \mathbb{Z}^\omega$

• a quantitative measure is a function $f : \mathbb{Z}^\omega \rightarrow \mathbb{R} \cup \{+\infty, -\infty\}$ that associates to each infinite sequence of integers a real value that summarizes the sequence

• we will consider the following measure:
  
  • $\text{Inf}$ that returns the infimum of the values along the sequence
  • $\text{Sup}$ that returns the supremum
  • $\text{Lim-Inf}$ that returns the inferior limit
  • $\text{Lim-Sup}$ that returns the superior limit
  • $\text{MP-Inf}$ that returns the inferior limit of the mean value
  • $\text{MP-Sup}$ that returns the superior limit of the mean value
  • $\text{Sum-Inf}$ that returns the inferior limit of the sum (energy games)
In a quantitative game with measure \( f \), we associate values to states as follows:

The **value for Player 1** from state \( s \) is equal to

\[
\text{Sup}_{\lambda_1 \in \Lambda_1} \text{Inf}_{\lambda_2 \in \Lambda_2} f(\text{Outcome}_A(s, \lambda_1, \lambda_2))
\]

i.e. the **supremum of the values** that Player 1 can force from \( s \) against all possible strategies of Player 2

Symmetrically, **the value for Player 2** from state \( s \) is equal to

\[
\text{Inf}_{\lambda_2 \in \Lambda_2} \text{Sup}_{\lambda_1 \in \Lambda_1} f(\text{Outcome}_A(s, \lambda_1, \lambda_2))
\]

A state \( s \) has **value** \( v \) if

\[
\text{Sup}_{\lambda_1 \in \Lambda_1} \text{Inf}_{\lambda_2 \in \Lambda_2} f(\text{Outcome}_A(s, \lambda_1, \lambda_2)) = \text{Inf}_{\lambda_2 \in \Lambda_2} \text{Sup}_{\lambda_1 \in \Lambda_1} f(\text{Outcome}_A(s, \lambda_1, \lambda_2)) = v
\]

A class of quantitative games for which each state has a value is said to be **determined**.
Value and threshold problems

• Given a weighted game arena $A$, a state $s$, a measure $f$, and a threshold value $v \in \mathbb{Q}$, the **threshold problem** asks

\[
\text{if Player 1 has a strategy } \lambda_1 \in \Lambda_1 \text{ such that } \\
\text{Outcome}_A(s, \lambda_1) \subseteq \{ \rho | f(\rho) \geq v \}
\]

or equivalently, if Player 1 has a strategy $\lambda_1 \in \Lambda_1$ such that for all strategies $\lambda_2 \in \Lambda_2$ of Player 2, $f(\text{Outcome}_A(s, \lambda_1, \lambda_2)) \geq v$

• The **value problem** asks to compute for each state $s \in S$ the value $v$ that Player 1 can force:

\[
v = \sup_{\lambda_1 \in \Lambda_1} \inf_{\lambda_2 \in \Lambda_2} f(\text{Outcome}_A(s, \lambda_1, \lambda_2))
\]

• **Remark.** The value problem can often be efficiently reduced to a series of queries to the threshold problem.
What is the value of each state for the \textbf{Inf} measure?
What is the value of each state for the Inf measure?

1:1, 2:1, 3:3, 4:2
Examples

What is the value of each state for the \textbf{Lim-Inf} measure?
Examples

What is the value of each state for the **Lim-Inf** measure?

1:2, 2:2, 3:3, 4:2
1 Dimensional
Inf, Sup, Lim-Inf, Lim-Sup
Games
Inf, Sup, Lim-Inf, Lim-Sup are regular measures

- Inf, Sup, Lim-Inf, Lim-Sup are regular measures in the following sense:

  Let \( v \in \mathbb{Q} \), the set of plays \( \rho \in \text{Plays}(A) \) s.t.

  \[
  f(\rho) \sim v
  \]

  for \( f \in \{\text{Inf, Sup, Lim-Inf, Lim-Sup}\} \) and \( \sim \in \{<,\leq,=,\geq,>\} \)

  is an omega-regular subset of \( S^\omega \)

- We will show how to reduce threshold problems for those measures to classes of omega-regular games that we have studied in the first part of this series of lectures.
Weighted edges transformation

Each edge is split using a new state with only one predecessor, the source of edge e, and only one successor, the target of edge e.

Arbitrarily, those states belong to Player 1.

Each play in A’ uniquely corresponds to a play in A and vice versa.
Sup games as reachability games

- Given a weighted game arena $A$ and a threshold $v \in \mathbb{Q}$, Player 1 has a strategy to win the objective

$$\{ \rho \in \text{Plays}(A) \mid \text{Sup}(\rho) \geq v \}$$

if he can force to reach a transition with value $w(e) \geq v$.

- To reduce the Sup threshold problem to a reachability problem, we apply the weighted edge transformation and consider the reachability objective $\text{Reach}\{ s_e \mid w(e) \geq v \}$, and so Sup threshold problems are linear time reducible to reachability game problems.

- Lemma. Player 1 has a strategy to win $\{ \rho \in \text{Plays}(A) \mid \text{Sup}(\rho) \geq v \}$ in $A$ iff Player 1 has a strategy to win the objective $\text{Reach}\{ s_e \mid w(e) \geq v \}$ in $A'$. 

Inf games as safety games

- Given a weighted game arena $A$ and a threshold $v \in \mathbb{Q}$, Player 1 has a strategy to win the objective $\{ \rho \in \text{Plays}(A) \mid \text{Inf}(|\rho|) \geq v \}$ if he can always avoid to take a transition $e$ with value $w(e) < v$.

- To reduce the Inf threshold problem to a safety game problem, we apply the weighted edge transformation and consider the safety objective $\text{Safe}(S' \{ s_e \mid w(e) < v \})$ in $A'$, so Inf threshold problems are linear time reducible to safety game problems.

- **Lemma.** Player 1 has a strategy to win $\{ \rho \in \text{Plays}(A) \mid \text{Inf}(|\rho|) \geq v \}$ in $A$ iff Player 1 has a strategy to win the safety objective $\text{Safe}(S' \{ s_e \mid w(e) < v \})$ in $A'$. 
Lim-Sup and Lim-Inf games

• Exercises:
  
  • Show how to reduce a Lim-Sup game to a Büchi game

  • Show how to reduce a Lim-Inf game to a coBüchi game
Computing the value of a state

• For the measures \textbf{Inf}, \textbf{Sup}, \textbf{Lim-Inf}, and \textbf{Lim-Sup}, the value that Player 1 can force in a state is equal to one of the values that appear on the edges of the weighted game arena.

• So, given a weighted game arena \( A \) with weight function \( w \), the set of potential values of a state \( s \) for Player 1 belongs to the finite set:

\[
\{ w(e) \mid e \in E \}
\]

• Now, for \textbf{Sup}, based on the reduction of the threshold problem to a reachability problem, we have that:

the value of \( s \) for Player 1 is the \textbf{maximal value} \( v \) s.t. Player 1 has a strategy to win \textbf{Reach}\{s \mid w(e) \geq v\} in \( A' \)

• As a direct consequence, the value of a state for Player 1 can be computed using a \textbf{binary search} and a logarithmic number of queries to the associated threshold problem.
Quantitative determinacy for Sup, Inf, Lim-Sup, Lim-Inf

**Theorem.** For all \( \mathbb{Z} \)-weighted game arena \( A \), for all states \( s \in S \), for all \( f \in \{ \text{Sup}, \text{Inf}, \text{Lim-Sup}, \text{Lim-Inf} \} \), we have that

\[
\text{Sup}_{\lambda_1 \in \Lambda_1} \text{Inf}_{\lambda_2 \in \Lambda_2} f(\text{Outcome}_A(s, \lambda_1, \lambda_2)) = \text{Inf}_{\lambda_2 \in \Lambda_2} \text{Sup}_{\lambda_1 \in \Lambda_1} f(\text{Outcome}_A(s, \lambda_1, \lambda_2)).
\]

**Proof.** This is a direct consequence of the determinacy results for reachability, safety, Büchi and coBüchi games.
1 Dimensional Mean-payoff and Energy Games
Mean-payoff games

\[ \rho = (1, 4) (4, 5) (5, 4) \ldots (4, 5) (5, 4) \ldots \]

\[ = \lim_{n \to +\infty} \frac{\sum_{i=1}^{n} r_i}{n} \]

\[ = \text{MP-Inf} \left( (1, 4) (4, 5) (5, 4) \ldots (4, 5) (5, 4) \ldots \right) = 1 \]

\[ \text{Win}_1 = \{ \text{play } \pi \mid \text{MP-Inf}(\pi) \geq v \} \]

Note: not \( \omega \)-regular.
Mean-payoff games

Win_1 = \{ \text{play} \in \pi | \text{MP-Inf}(\pi) \geq v \}

Note: not \( \omega \)-regular.

W.l.o.g., \( v=0 \)

\( \rho=(1,4) \ (4,5) \ (5,4) \ldots \ (4,5) \ (5,4) \ldots = \text{play} \)

\[
\frac{1}{n} \sum_{i=1}^{n} r_i \rightarrow \text{Lim-Inf}_{n \rightarrow +\infty} \sum_{i=1, i=n} r_i / n
\]

= MP-Inf((1,4) \ (4,5) \ (5,4) \ldots \ (4,5) \ (5,4)\ldots)=1
Mean-payoff games

$$\text{MP-Sup} = \limsup_{n \to +\infty} \frac{\sum_{i=1}^{n} r_i}{n}$$

$$\rho = (1,4) (4,5) (5,4) \ldots (4,5) (5,4) \ldots$$

$$= \liminf_{n \to +\infty} \frac{\sum_{i=1}^{n} r_i}{n}$$

$$= \text{MP-Inf}((1,4) (4,5) (5,4) \ldots (4,5) (5,4)\ldots) = 1$$

The distinction between MP-Inf and MP-Sup is **not** important in the **one-dimensional** case.

We will come back to the distinction when considering the **multi-dimensional** case.
Mean-payoff games

Win$_1$ = \{ play $\pi$ | $\text{MP}(\pi) \geq v$ \}

Note: not $\omega$-regular.
Energy games

Edges are labelled with energy consumptions or energy gains.

For EG the take the sum and not the mean.

Initial energy level : 7

Play : (1,2) (2,1) (1,4) (4,5) (5,4) (4,5) (5,4) ...

EL : 7 8 3 7 10 9 12 11 ...

$\models \Box EL(7) \geq 0$
Energy games

Edges are labelled with energy consumptions or energy gains.

Initial energy level: 7

Play: (1,2) (2,1) (1,4) (4,5) (5,4) (4,5) (5,4) ...

EL: 7 8 3 7 10 9 10 9 ...

⊨ ☐ EL ≥ 0
Mean-payoff and energy games

The **threshold problem for mean-payoff games** asks: given a state $s$, if Player 1 has a strategy $\lambda_1$ s.t.

$$\text{Outcome}_A(s,\lambda_1) \subseteq \text{Win}_1 = \{ \text{plays } \rho \mid \text{MP}(\rho) \geq 0 \}$$

The decision problem for energy games, called **the unknown initial credit problem**, asks: given state $s$, decide if there exist

1. an **initial energy level** $c_0 \in \mathbb{N}$, and
2. a strategy $\lambda_1$ for Player 1 to maintain a positive energy level from $c_0$ at all time: $\text{Outcome}_A(s,\lambda_1) \models \Box \text{EL}(c_0) \geq 0$

Note that this is equivalent to ask if there exists a strategy $\lambda_1$ for Player 1 such that for all $\rho \in \text{Outcome}_A(s,\lambda_1)$ the **infimum of the sum** along the prefixes is bounded from below (not equal to $-\infty$)
Determinacy and Equivalence for MPG and EG
An Elementary Proof
Determinacy of MPG

Theorem [Determinacy] For all MPG G, for all states s:
• either $\exists \lambda_1$ for Player 1 s.t. $\text{Outcome}(s, \lambda_1) \subseteq \{ \pi | \text{MP}(\pi) \geq 0 \}$,
• or $\exists \lambda_2$ for Player 2 s.t. $\text{Outcome}(s, \lambda_2) \subseteq \{ \pi | \text{MP}(\pi) < 0 \}$.

Proof. An elementary proof can be obtained from FCU and Zermelo’s determinacy theorem.
Zermelo's theorem

Theorem [Zermelo 1913]. Every finite tree (reachability) game is determined.

- Every finite duration turn-based game can be represented as a game tree of bounded depth
- Each branch represents a play
- The winning condition is defined by a partition of the leaves of the tree: plays that are winning for Player 1 and those that are winning for Player 2
- A corollary: in chess, either black or white (one of the two players) is able to force win or draw
FCU for solving MPG

**FCU**: unfold the weighted graph up to a first repetition of vertex:
- a leaf is **winning for Pl. 1** if the cycle has a non-negative sum
- a leaf is **winning for Pl. 2** if the cycle has a negative sum

♫ By Zermelo's theorem:
either Pl. 1 can force **non-negative cycles**
or Pl. 2 can force **negative cycles**
Unfolding - an example

Unfolding of $A$ starting in $1$

$w(cycle) = 1$

$w(cycle) = -1$

$w(cycle) = 1$
Unfolding - an example

In this example, Player 1 can force non-negative cycles!
Transfer of strategies

MPG-EG

Lemma [strategy transfer] Winning strategies in the FCU can be transferred into winning strategies in the MP/EG game:

- If Player 1 can force green leaves in the unfolding of A then Player 1 has a winning strategy in A for the MP \( \geq 0 \) objective and in the EG;

- If Player 2 can force red leaves in the unfolding of A then Player 2 has a winning strategy in A for the MP \( < 0 \) objective and in the EG.

To establish this lemma, we rely on the notion of cycle decomposition of a play…

… and we will get:

**Corollary [MPG≈EG]** MPG A •\( \geq v \) and EG A-v are equivalent!
Player 1 ensures MP value 1 in A \textit{iff} Player 1 wins EG A-1.
Transfer of strategies

Let \( \lambda_1 \) be a winning strategy for Player 1 in the \( \text{FCU}(A) \). Let \( \lambda_1^* \) be the transfer of this strategy in \( A \). Then all (simple) cycles obtained during the cycle decomposition of any outcome of \( \lambda_1^* \) in the game arena \( A \) have sum of weights \( \geq 0 \).

So, the running sum of all prefixes is bounded from below by \(-nW\) (\( n=\)number of states in \( A \), \( W=\)absolute value of the largest weight in \( A \), i.e. \(-nW\) is a bound on the maximal negative value of the residue along the cycle decomposition)

- This implies that the \( \text{EG} \) is won by Player 1 from initial energy level \( nW \)
- The \( \text{MP} \) of the play is non-negative, so Player 1 wins \( \text{MP} \geq 0 \)
Transfer of strategies

Let \( \lambda_2 \) be a winning strategy for Player 2 in the \( \text{FCU}(A) \). Let \( \lambda_2^* \) be the transfer of this strategy in A. Then all (simple) cycles obtained during the cycle decomposition of any outcome of \( \lambda_2^* \) in the game arena A have sum of weights < 0.

So, in this case, the running sum of the prefixes tends to -\( \infty \) and each cycle has a MP \( \leq -1/n \)

- The energy game is won by Player 2 no matter what is the initial energy level
- The MP of the play is \( \leq -1/n \) (the finite residue on the stack can be neglected in the long run) and the MP is won by Player 2
Determinacy and equivalence of MPG-EG

**Theorem [MPG strong determinacy]** For all $\mathbb{Z}$-weighted game arena $A$, for all states $s$:
- either $\exists \lambda_1$ for Player 1 s.t. $\text{Outcome}_A(s, \lambda_1) \subseteq \{ \rho \mid MP(\rho) \geq 0 \}$,
- or $\exists \lambda_2$ for Player 2 s.t. $\text{Outcome}_A(s, \lambda_2) \subseteq \{ \rho \mid MP(\rho) \leq -1/n \}$.

**Corollary [MPG determinacy]** For all $\mathbb{Z}$-weighted game arena $A$, for all states $s$:
- either $\exists \lambda_1$ for Player 1 s.t. $\text{Outcome}_A(s, \lambda_1) \subseteq \{ \rho \mid MP(\rho) \geq 0 \}$,
- or $\exists \lambda_2$ for Player 2 s.t. $\text{Outcome}_A(s, \lambda_2) \subseteq \{ \rho \mid MP(\rho) < 0 \}$.

We will state and establish a **quantitative determinacy** result later.
Determinacy and equivalence of MPG-EG

**Theorem [Determinacy-EG]** For all \( \mathbb{Z} \)-weighted game arena A, for all states \( s \):
- either there exists an initial EL and a strategy for Player 1 from \( s \) to win EG,
- or Player 2 has a strategy from \( s \) to win the EG, no matter what is the initial EL.

**Theorem [Equivalence MPG-EG]** For all \( \mathbb{Z} \)-weighted game arena A, for all states \( s \), Player 1 wins for \( MP \geq 0 \) from \( s \) if and only if Player 1 wins the EG from \( s \).
Memoryless determinacy

Choice in 2 is not uniform: it depends on the history =need for memory

‼ Strategies in the tree are **not** guaranteed to correspond to **memoryless** strategies in the graph!
‼ We need additional arguments to prove memoryless determinacy of MPG and EG …
The arguments that we need are based on a fixed point algorithm for solving energy games.
A Pseudo-Polynomial Time Algorithm for EG (fixpoint algorithm)
Upward closed sets in $\mathbb{N}^k$

• Let $v_1, v_2 \in \mathbb{N}^k$, we write $v_1 \leq v_2$ when for all $i$, $1 \leq i \leq k$, $v_1(i) \leq v_2(i)$.

• A subset $U \subseteq \mathbb{N}^k$ is upward closed if for all $v_1, v_2 \in \mathbb{N}^k$ if $v_1 \in U$ and $v_1 \leq v_2$ then $v_2 \in U$.

• Given $U \subseteq \mathbb{N}^k$, we define $\uparrow U = \{ v' \in \mathbb{N}^k \mid \exists v \in U \text{ and } v \leq v' \}$ and call it the upward closure of $U$. 
EG and safety games

SAFE$_i$ = set of $(s, c) \in S_1 \times \mathbb{N} \cup S_2 \times \mathbb{N}$ s.t. from $(s, c)$, Player 1 can maintain energy level non-negative for $i$ steps

What are the controllable predecessors of SAFE$_i$?

SAFE$_i$ is an upward closed set
We define $\preceq$ as $(s,c) \preceq (s',c')$ iff $s = s'$ and $c \leq c'$

- CPRE$_1(X)$ transforms $\preceq$-upward closed sets into $\preceq$-upward closed sets

\[
\text{CPRE}_1(X) \text{ where } X \subseteq (S_1 \times \mathbb{N} \cup S_2 \times \mathbb{N}) \text{ is the set}
\]
\[
\{ (s_1,c) \in S_1 \times \mathbb{N} \mid \exists (s_1,w,s') \in E : (s',c+w) \in X \} 
\]
\[
\cup \{ (s_2,c) \in S_2 \times \mathbb{N} \mid \forall (s_2,w,s') \in E : (s',c+w) \in X \}
\]

• We define $\preceq$ as $(s,c) \preceq (s',c')$ iff $s = s'$ and $c \leq c'$

• CPRE$(X)$ transforms $\preceq$-upward closed sets into $\preceq$-upward closed sets
$\text{CPRE}_{1}(X)$

$\text{CPRE}_{1}(X)$ where $X \subseteq (S_1 \times \mathbb{N} \cup S_2 \times \mathbb{N})$ is the set

\[
\{ (s_1,c) \in S_1 \times \mathbb{N} \mid \exists (s_1,w,s') \in E : (s',c+w) \in X \} \\
\cup \\
\{ (s_2,c) \in S_2 \times \mathbb{N} \mid \forall (s_2,w,s') \in E : (s',c+w) \in X \}
\]

$\text{SAFE}_0 = \uparrow \{(s_1,0),(s_2,0),(s_3,0),(s_4,0)\}$
$\text{SAFE}_1 = \uparrow \{(s_1,0),(s_2,2),(s_3,0),(s_4,1)\}$
$\text{SAFE}_2 = \uparrow \{(s_1,0),(s_2,2),(s_3,0),(s_4,2)\}$
$\text{SAFE}_3 = \uparrow \{(s_1,0),(s_2,2),(s_3,0),(s_4,3)\}$
$\text{SAFE}_4 = \uparrow \{(s_1,0),(s_2,2),(s_3,0),(s_4,4)\}$
$\text{SAFE}_5 = \uparrow \{(s_1,0),(s_2,2),(s_3,0),(s_4,5)\}$
$\text{SAFE}_6 = \uparrow \{(s_1,1),(s_2,2),(s_3,0),(s_4,6)\}$
$\text{SAFE}_7 = \uparrow \{(s_1,2),(s_2,2),(s_3,0),(s_4,7)\}$
$\text{SAFE}_8 = \uparrow \{(s_1,3),(s_2,2),(s_3,0),(s_4,8)\}$
$\text{SAFE}_9 = \uparrow \{(s_1,3),(s_2,2),(s_3,0),(s_4,9)\}$

$\ldots$

$\text{SAFE}_k = \uparrow \{(s_1,3),(s_2,2),(s_3,0),(s_4,k)\}$ no stabilisation!
CPRE$_1[C](X)$ to force termination

- Above energy requirement $C \in \mathbb{N}$, we consider the game as lost! (conservative approximation)

- Let $C \in \mathbb{N}$, define $U(C) = \mathcal{P}((S_1 \cup S_2) \times \{0\ldots C\})$

- $\text{CPRE}_1[C](X)$ where $X \in U(C)$ is the set

  \[
  \{ (s_1,c) \in S_1 \times \{0\ldots C\} \mid \exists (s_1,w,s') \in E : (s',c+w) \in X \} \\
  \cup \quad \{ (s_2,c) \in S_2 \times \{0\ldots C\} \mid \forall (s_2,w,s') \in E : (s',c+w) \in X \}
  \]
CPRE$_1[\mathbf{C}](X)$ - properties

CPRE$^*[\mathbf{C}](X)$ is monotone over $\mathcal{P}((S_1 \cup S_2) \times \{0\ldots C\})$

• so it has a greatest fixed point, noted CPRE$^*[\mathbf{C}]$

• computed iteratively from $T = \mathcal{P}((S_1 \cup S_2) \times \{0\ldots C\})$

• convergence is ensured now as $\mathcal{P}((S_1 \cup S_2) \times \{0\ldots C\})$ is finite

• The greatest fixpoint computed in $O(|V|.|E|.W)$, where $W$ is the largest weight in absolute value in arena $A$: the complexity is pseudo-polynomial
\( \text{SAFE}_0 = \uparrow \{ (s_1,0), (s_2,0), (s_3,0), (s_4,0) \} \)

\( \text{SAFE}_1 = \uparrow \{ (s_1,0), (s_2,2), (s_3,0), (s_4,1) \} \)

\( \text{SAFE}_2 = \uparrow \{ (s_1,0), (s_2,2), (s_3,0), (s_4,2) \} \)

\( \text{SAFE}_3 = \uparrow \{ (s_1,0), (s_2,2), (s_3,0), (s_4,3) \} \)

\( \text{SAFE}_4 = \uparrow \{ (s_1,3), (s_2,2), (s_3,0) \} \)

\( \text{SAFE}_5 = \uparrow \{ (s_1,3), (s_2,2), (s_3,0) \} = \text{SAFE}_\infty \)

\( \text{CPRE}_1[3](X) \) where \( X \in U(C) \) is the set
\[
\{ (s_1,c) \in S_1 \times \{0,..,3\} \mid \exists (s_1,w,s') \in E : (s',c+w) \in X \}\cup
\{ (s_2,c) \in S_2 \times \{0,..,3\} \mid \forall (s_2,w,s') \in E : (s',c+w) \in X \}
\]

\[ \text{G} \]

Stabilisation !

Greatest fixpoint
Theorem \textbf{correctness} \( \forall C \in \mathbb{N}, \forall (s,c) \in \uparrow \text{CPRE}^*_1[C], \) Player 1 wins EG from \( s \).
Correctness

**Theorem [correctness]** \( \forall C \in \mathbb{N}, \forall (s,c) \in \uparrow \text{CPRE}_1^*[C], \) Player 1 wins EG from s.

**Proof.** Assume we start in state s with energy level c. We construct a strategy for Player 1 s.t. if \((s_1,c_1)(s_2,c_2),\ldots,(s_n,c_n),\ldots\) is an outcome then for all positions \(i \geq 0,\) \((s_i,c_i) \in \uparrow \text{CPRE}^*[C].\) So the energy level always stays non-negative.

The proof is by induction. We consider two cases.

**1.** \((s_{i-1},c_{i-1}) \in \uparrow \text{CPRE}_1^*[C]\) and \(s_{i-1} \in S_1,\) consider \((s,c) \in \text{CPRE}_1^*[C],\) with \(s_{i-1}=s\) and \(c_{i-1} \geq c.\) From \((s_{i-1},c_{i-1})\) Player 1 chooses \(e=(s,w,s') \in E\) such that there exists \((s',c') \in \text{CPRE}_1^*[C]\) such that \(c+w \geq c'.\) As \(\text{CPRE}_1^*[C]\) is a FP, such an edge exists. So, we have that \((s_i,c_i)\) is such that \(s_i=s',\ c_i=c_{i-1}+w \geq c+w \geq c'\) and so \((s_i,c_i) \in \uparrow \text{CPRE}_1^*[C].\)

**2.** \((s_{i-1},c_{i-1}) \in \uparrow \text{CPRE}_1^*[C]\) and \(s_{i-1} \in S_2,\) and assume that Player 2 has chosen the edge \((s_{i-1},w,s_i).\) Let \((s,c) \in \text{CPRE}_1^*[C]\) be s.t. \(s_{i-1}=s\) and \(c_{i-1} \geq c.\) By definition of \(\text{CPRE}_1^*[C],\) there exists \((s_i,c') \in \text{CPRE}_1^*[C]\) such that \(c+w \geq c'.\) So, we have that \(c_i=c_{i-1}+w \geq c+w \geq c'\) and we are done.
Completeness

**Theorem [completeness]** Let $C=2nW$. If Player 1 has a winning strategy from $s_0$ in EG, then there exists $(s_0, c) \in \text{CPRE}_1^*[C]$.

Claim:
$F=\uparrow\{(s_1,20),(s_2,19),(s_3,17)\}$ is a FP of $\text{CPRE}_1[20]$
So, $F \subseteq \text{CPRE}_1^*[20]$!
Completeness

**Theorem [completeness]** Let \( C = 2nW \). If Player 1 has a winning strategy from \( s_0 \) in EG, then there exists \((s_0, c) \in \text{CPRE}_1^* [C] \).

**Proof.** Consider the first cycle unfolding of \( A \) from \( s_0 \) and the associated reachability game (i.e. to reach leaves associated to non-negative cycles). If Player 1 wins the EG from \( s_0 \), then Player 2 cannot win the tree reachability game (because of strategy transfer), and so by determinacy Player 1 has a winning strategy in the tree.

Consider any strategy of Player 1 in the tree and the subtree induced by that strategy. We annotate the subtree as follows. The root is labelled with weight \( nW \). Then we label the other nodes starting from the root by maintaining the energy level on each history.

It is easy to see that this tree only contains energy levels \( c \) such that \( 0 \leq c \leq 2nW \), indeed each branch is of length at most \( n \) and so from energy level \( nW \), we can gain at most \( nW \) and lose at most \( nW \).

Let \( F = \{ (s, c) \in S \times \{ 0, \ldots, 2nW \} \mid \exists \text{ node } \pi \text{ in the tree labelled with } s \text{ and } c' \text{ and } c \geq c' \} \). Clearly, \((s_0, nW) \in F\), and F is a fix point for the operator \( \text{CPRE}[2nW] \). So \( F \subseteq \text{CPRE}_1^* [2nW] \) (as \( \text{CPRE}_1^* [2nW] \) is the greatest fixpoint), and \((s_0, nW) \in \text{CPRE}_1^* [2nW] \).
Memoryless strategies for Player 1 in EG

Fixpoint and good edges for Player 1

\( \text{CPRE}^*[C] \)
Memoryless strategies for Player 1 in EG

Fixpoint and good edges for Player 1

Important property:

edges that are good for EL c are also good for all EL c’ > c

= Monotonicity implies Memoryless

CPRE*[C]
Memoryless strategies for Player 1 in EG

- Player 1 wins the EG from \( W_1 = \{ s | \exists (s, c) \in \text{CPRE}_1^*[2nW] \} \).
- For each \( s \in W_1 \cap S_1 \), consider \( (s, c) \) where \( c \) is minimal (worst-case situation) in \( \text{CPRE}_1^*[2nW] \).
- From each minimal pair \( (s, c) \), fix an edge \( (s, w, s') \) such \( (s, c+w) \in \uparrow \text{CPRE}_1^*[2nW] \).

**Theorem [memoryless].** Strategy \( \lambda_1 \) is a memoryless (uniform) strategy \( \lambda_1 \) which is winning from all Player 1 winning states of \( A \) for the energy objective.
Memoryless strategies for Player 1 and 2 in MPG

As a corollary of MPG≈EG, and strong determinacy of MPG, we get:

**Theorem [memoryless determinacy of MPG]**
Mean-payoff games are memoryless determined, i.e. both Player 1 and Player 2 can play optimally with memoryless strategies.

**Proof.** Player 1 can play memoryless as it is the case in EG. For Player 2, we do the following reasoning: Player 2 can enforce MP ≤ -1/n in A if and only if Player 2 can enforce MP ≥ 0 in A’ where A’ is equal to A but the weight w in A is replaced by -w-1/n in A’. So Player 2 can play optimally with a memoryless in A.

Now, we obtain the following corollary for EG:

**Corollary [memoryless strategies for Player 2 in EG].**
If Player 2 wins EG from s then he has a memoryless winning strategy from s.
The complexity of MPG and EG

As a direct corollary of the memoryless determinacy of MPG and EG we obtain:

**Theorem.** The threshold problem for mean-payoff game can be solved in $\text{NP} \cap \text{coNP}$.

**Theorem.** The unknown initial energy problem for EG can be solved in $\text{NP} \cap \text{coNP}$.

• While we have pseudo-polynomial time algorithms to solve those problems, we do not have proper polynomial time algorithms so far.