Concurrent parameterized games

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1 Introduction

Traditional concurrent games on graphs [1] involve a fixed number of players, who take decisions simultaneously, determining the next state of the game. In our work [2], we introduce a parameterized variant of concurrent games on graphs, where the parameter is precisely the number of players. Parameterized concurrent games are described by finite graphs, in which the transitions bear regular languages to describe the possible move combinations that lead from one vertex to another.

As for traditional concurrent games, one can consider natural questions such as, for instance, the distributed synthesis problem, or the existence and computation of Nash equilibria etc. To start with, we consider a simpler decision problem: the first player, called Eve, is distinguished, and the question is whether she can ensure a reachability objective against the coalition of the other players, not knowing *a priori* the number of her opponents. She therefore must play uniformly, whatever the number of opponents she has.

2 Game setting

We first introduce a simpler setting, where we assume that the languages on transitions of the arena are particularly simple: they only constrain the number of opponents Eve has. We later show this simpler setting is not restrictive for the decision problem with arbitrary regular languages.

Definition 1. *A* parameterized arena *is a tuple $\mathcal{A} = \langle V, \Sigma, \Delta \rangle$ where V is a finite set of vertices; Σ is a finite set of actions; and $\Delta : V \times \Sigma \times \mathbb{N}_{>0} \to 2^V$ is the transition function.*

The arena is *deterministic* if for every $v \in V$, and every pair $(a, k) \in \Sigma \times \mathbb{N}_{>0}$, there is at most one vertex $v' \in V$ such that $v' \in \Delta(v, a, k)$. Action $a \in \Sigma$ is *enabled* at vertex v if there exists $k \in \mathbb{N}_{>0}$ such that $\Delta(v, a, k) \neq \emptyset$. The arena is assumed to be *complete for enabled actions*: for every $v \in V$, if a is enabled at v, then for all $k \in \mathbb{N}_{>0}$, $\Delta(v, a, k) \neq \emptyset$. This assumption is natural: Eve does not know how many opponents she has, and the successor vertex must exist whatever that number is. Given a predicate $P \subseteq \mathbb{N}_{>0}$, $\Delta(v, a, P)$ is a shorthand for $\bigcup_{k \in P} \Delta(v, a, k)$.

Further, for any $v, v' \in V$ and $a \in \Sigma$, we introduce the following notation to represent the set of number of opponents that can lead from v to v' under action a of Eve: $\nabla(v, a, v') = \{k \in \mathbb{N}_{>0} \mid v' \in \Delta(v, a, k)\}$. Finally, we write $E = \{(v, a, v') \mid \exists k \in \mathbb{N}_{>0},\ v' \in \Delta(v, a, k)\}$ for the set of *edges* of the arena.

Let $k \in \mathbb{N}_{>0}$. A *k-history*, for a coalition composed of k opponents of Eve, is a finite sequence $v_0 a_0 \cdots v_i \in (V \cdot \Sigma)^* \cdot V$ such that for every $j < i$, $v_{j+1} \in \Delta(v_j, a_j, k)$ (or equivalently $k \in \bigcap_{j<i} \nabla(v_j, a_j, v_{j+1})$). A *history* in $\mathcal{A}$ is a k-history for some $k \in \mathbb{N}_{>0}$. We note $\mathsf{Hist}(k)$ (resp. Hist) for the set of k-histories (resp. histories) in $\mathcal{G}$. Similar notions of a *k-play* and a *play* are defined for infinite sequences.

Definition 2. *A* strategy *for Eve from v in $\mathcal{A}$ is a mapping $\sigma : \mathsf{Hist} \to \Sigma$ that associates to every history $hv' \in \mathsf{Hist}$ an action $\sigma(hv')$ which is enabled at v'. Further, σ is memoryless whenever for every $hv', h'v' \in \mathsf{Hist}$, $\sigma(hv') = \sigma(h'v')$.*

A strategy for Eve is applied with no prior information on the number of her opponents. Given a strategy σ, an initial vertex v and $k \in \mathbb{N}_{>0}$ a number of opponents, we define the outcome $\mathsf{Out}(\sigma, v, k)$ as the set of plays that σ induces from v when Eve has exactly k opponents. Formally, $\mathsf{Out}(\sigma, v, k)$ is the set of all k-plays $\rho = v_0 a_0 v_1 a_1 v_2 \cdots$ such that $v = v_0$, and for all $i \geq 0$, $\sigma(v_0 a_0 \cdots v_i) = a_i$ and $v_{i+1} \in \Delta(v_i, a_i, k)$. The completeness assumption ensures that the set $\mathsf{Out}(\sigma, v, k)$ is not empty. Finally, $\mathsf{Out}(\sigma)$ is the set of all possible plays induced by σ from v: $\mathsf{Out}(\sigma, v) = \bigcup_{k \geq 1} \mathsf{Out}(\sigma, v, k)$.

Given an arena $\mathcal{A} = \langle V, \Sigma, \Delta \rangle$, a target vertex $t \in V$ defines a *reachability game* $\mathcal{G} = (\mathcal{A}, t)$ for Eve. A strategy σ for Eve from v in the reachability game $\mathcal{G} = (\mathcal{A}, t)$ is *winning* if all plays in

$\mathsf{Out}(\sigma, v)$ eventually reach t. If there exists a winning strategy from v, then we say that v belongs to the *winning region* of Eve.
The purpose of our work was to establish the complexity of the following decision problem:

PARAMETERIZED REACHABILITY GAME PROBLEM **Input**: A parameterized reachability game $\mathcal{G} = (\mathcal{A}, t)$ and an initial vertex v. **Question**: Does Eve have a winning strategy from v in $\mathcal{G}$?

For algorithmic reasons, we assume the transition function Δ of $\mathcal{A}$ can be described in a finite way. More precisely, the sets $\nabla(v, a, v')$ for $v, v' \in V$ and $a \in \Sigma$ should be simple enough.

We first consider constraints described by closed intervals or finite unions of closed intervals. As a complexity parameter, we use $\#\mathsf{endpoints}_{\mathcal{A}}$, the number of endpoints used in constraints in $\mathcal{A}$. All the complexities will be functions of this parameter, independently of the precise values of the endpoints. More generally, we also consider semilinear predicates over $\mathbb{N}$. W.l.o.g. we assume semilinear sets are given as finite unions of ultimately periodic sets of integers. A set $S \subseteq \mathbb{N}$ is *ultimately periodic* if there exist a threshold $t \in \mathbb{N}$ and a period $p \in \mathbb{N}$ such that for all $a, b \in \mathbb{N}$ with $a, b \geq t$ and $a \equiv b \mod p$, we have $a \in S$ iff $b \in S$. For complexity issues, all constants are assumed to be represented in binary. In that context, as a complexity parameter, we use $\#\mathsf{pred}_{\mathcal{A}}$, the number of predicates used on edges of $\mathcal{A}$.

3 Complexity results

From a parameterized reachability game, we construct a standard two-player turn-based game, called the *knowledge game*, which precisely captures the partial-information of Eve in the parameterized game. We will show that existence of a winning strategy for Eve in the parameterized game reduces to the resolution of the knowledge game. However the knowledge game can be a priori exponential in the size of the original arena, and this exponential blowup is unavoidable. Yet, later we will show the parameterized reachability game problem can be solved in polynomial space in the most general case, when constraints are semilinear predicates; moreover for simpler constraints, we show the problem to be complete for smaller complexity classes.

The knowledge game. From a parameterized reachability game, we construct a turn-based two-player *knowledge game* as follows.

Definition 3. *Let $\mathcal{G} = (\mathcal{A}, t)$ be a parameterized game, with $\mathcal{A} = \langle V, \Sigma, \Delta \rangle$. The* knowledge game *associated with $\mathcal{G}$ is the two-player turn-based reachability game $\mathcal{K}_{\mathcal{G}} = (V_E \cup V_A, \Delta_{\mathcal{K}}, F)$, between Eve and Adam, such that $V_E \subseteq V \times 2^{\mathbb{N}_{>0}}$ and $V_A \subseteq V_E \times \Sigma$ are Eve and Adam vertices, respectively; $\Delta_{\mathcal{K}} \subseteq (V_E \times V_A) \cup (V_A \times V_E)$ is the edge relation; and $F = V_E \cap \{(t, K) \mid K \subseteq \mathbb{N}_{>0}\}$ is the set of target vertices. They are defined inductively by*

- $\{(v, \mathbb{N}_{>0}) \mid v \in V\} \subseteq V_E$;
- $\forall (v, K) \in V_E$, $\forall a \in \Sigma$ *enabled at* v, $(v, K, a) \in V_A$ *and* $\big((v, K), (v, K, a)\big) \in \Delta_{\mathcal{K}}$;
- $\forall (v, K, a) \in V_A$, $\forall v' \in V$ *such that* $K \cap \nabla(v, a, v') \neq \emptyset$, $(v', K \cap \nabla(v, a, v')) \in V_E$ *and* $\big((v, K, a), (v', K \cap \nabla(v, a, v'))\big) \in \Delta_{\mathcal{K}}$;

A *strategy* for Eve in $\mathcal{K}_{\mathcal{G}}$ is a function $\lambda : (V_E \cdot V_A)^* \cdot V_E \to V_A$ compatible with $\Delta_{\mathcal{K}}$. We borrow standard notions of outcomes and winning strategies from the literature.

It is not hard to see that the game $\mathcal{K}_{\mathcal{G}}$ is finite. Indeed, one can show by induction that every Eve's vertex (v, K) (hence every Adam's vertex (v, K, a)) is such that K is an intersection of finitely many sets of the form $\nabla(v', a, v'')$ or $\mathbb{N}_{>0}$.
One can show the correctness of the knowledge game construction:

Theorem 4 ([2]). *Eve has a winning strategy σ from v_0 in $\mathcal{G}$ if and only if she has a winning strategy λ from $(v_0, \mathbb{N}_{>0})$ in $\mathcal{K}_{\mathcal{G}}$.*

The following lemma states how the size of the knowledge game depends on the input arena:

Lemma 5. *For $\mathcal{G} = (\mathcal{A}, t)$ a parameterized game with $\mathcal{A} = \langle V, \Sigma, \Delta \rangle$, the size of the associated knowledge game $\mathcal{K}_{\mathcal{G}}$ is polynomial in $|V|$, $|\Sigma|$, and (1) exponential in $\#\mathsf{pred}_{\mathcal{A}}$, for constraints defined by semilinear predicates; (2) exponential in $\#\mathsf{endpoints}_{\mathcal{A}}$, for constraints defined by finite unions of intervals; and (3) polynomial in $\#\mathsf{endpoints}_{\mathcal{A}}$, for constraints defined as intervals. Furthermore, the exponential blowup is unavoidable in the two first cases.*

We then study upper and lower bounds for the parameterized reachability game problem depending on the presentation of the constraints in the input.

Theorem 6 ([2])**.** *The complexity of the parameterized reachability game problem is stated in Table 1.*

		Deterministic arenas	Non-deterministic arenas
Constraints	Intervals	PTIME-complete	
	Finite unions of intervals	NP-complete	PSPACE-complete
	Semilinear sets	PSPACE-complete	

Table 1: Complexity of the parameterized reachability game problem.

These complexities, when the constraints are given as (finite unions of) intervals, are in $\#\mathsf{endpoints}_{\mathcal{A}}$ used in the constraints but independent of values of the endpoints. On the other hand, when constraints are given as semilinear sets, the complexity depends on $\#\mathsf{pred}_{\mathcal{A}}$ as well as the size of the encodings of the semilinear sets.

4 Discussion: Beyond the number of players

Our model of parameterized game defined in Section 2, with constraints on the number of opponents for Eve, is actually a simplification of a general concurrent game model, where the transitions bear regular languages to describe the possible move combinations that lead from one vertex to another, which was mentioned in the introduction. Here we discuss that the latter in fact reduces to the simpler one.

Definition 7. *A* language-based parameterized arena *is a tuple* $\mathcal{A}_{\mathrm{L}} = \langle V, \Sigma, \Delta_{\mathrm{L}} \rangle$ *where* V *is a finite set of vertices;* Σ *is a finite set of actions; and* $\Delta_{\mathrm{L}} : V \times \Sigma^{\geq 2} \to 2^V$ *is the transition function.*

The fact that Eve has at least one opponent explains the term $\Sigma^{\geq 2}$ in the transition function. We assume that for every $(v, v') \in V^2$, $\nabla_{\mathrm{L}}(v, v') \stackrel{\text{def}}{=} \{w \in \Sigma^{\geq 2} \mid v' \in \Delta_{\mathrm{L}}(v, w)\}$ is regular.

The game is then played as follows, when $k{+}1$ is the number of players, called Eve, Adam_1, ..., Adam_k: from vertex v, each of the players select simultaneously an action in Σ; concatenating all the letters (Eve first, and then all Adams' actions), it forms a word w; the next vertex of the game is then one of the vertices v' in $\Delta_{\mathrm{L}}(v, w)$; the game then resumes from vertex v'. Strategies for Eve, and outcomes can be defined similarly to that of parameterized arenas in Section 2. The language-based parameterized game problem is then to decide whether Eve has a strategy that is winning against any number of opponents.

Language-based parameterized arenas generalize parameterized arenas: one can for instance replace rules of the form $v' \in \Delta(v, a, k)$ in a parameterized arena by $v' \in \Delta_{\mathrm{L}}(v, a\Sigma^k)$ to construct a language-based parameterized arena, preserving the winning region for Eve. For our problem of existence of a winning strategy for Eve, the reduction in the other direction also holds:

Proposition 8 ([2])**.** *The language-based parameterized reachability game problem reduces in polynomial time to the parameterized reachability game (with semilinear predicates).*

Thanks to Proposition 8, and using results from Theorem 6, we obtain the precise complexity of the language-based parameterized reachability game problem:

Theorem 9 ([2])**.** *The language-based parameterized reachability game problem is* PSPACE*-complete.*

References

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[2] N. Bertrand, P. Bouyer, and A. Majumdar. Concurrent parameterized games. In *39th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2019, December 11-13, 2019, Bombay, India*, volume 150 of *LIPIcs*, pages 31:1–31:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.