Guaranteed phase synchronization of hybrid oscillators using symbolic Euler's method

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Abstract

The phenomenon of *phase synchronization* was evidenced in the 17th century by Huygens while observing two pendulums of clocks leaning against the same wall. This phenomenon has more recently appeared as a widespread phenomenon in nature, and turns out to have multiple industrial applications. The exact parameter values of the system for which the phenomenon manifests itself are however delicate to obtain in general, and it is interesting to find formal sufficient conditions to *guarantee* phase synchronization. Using the notion of *reachability*, we give here such a formal method. More precisely, our method selects a portion S of the state space, and shows that any solution starting at S *returns* to S within a fixed number of periods T. Besides, our method shows that the components of the solution are then (almost) in phase. We explain how the method applies on the Brusselator reaction-diffusion example.

1 Showing synchronization using a reachability method

We consider a system composed of n subsystems governed by a system of differential equations (ODEs) of the form $\dot{x}(t) = f(x(t))$. For the sake of simplicity, we suppose here $n = 2$. The system of ODEs is thus of the form: $\dot{x_1}(t) = f_1(x_1(t), x_2(t))$ and $\dot{x_2}(t) = f_2(x_1(t), x_2(t))$ with $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^m \times \mathbb{R}^m$, where m is the dimension of the state space of each subsystem. The initial condition is of the form $(x_1^0, x_2^0) \in \mathbb{R}^m \times \mathbb{R}^m$.

The set $S = S_1 \times S_2$ (with $S_i \subset \mathbb{R}^m$, $i = 1, 2$) on which we focus our analysis, is selected as follows. We first consider, for each subsystem i ($i = 1, 2$), a "ring" of reduced width e_i around the cyclic trajectory (orbit). We then select a fragment of each ring, which gives two sets of states S_1 and S_2. Typically, for $i = 1, 2$, S_i is a *parallelogram* with a *horizontal* "base" of width e_i. The set S_i is thus characterized by a triple (a_i, b_i, e_i) where a_i and b_i are the end points of its main diagonal, and e_i the size of its horizontal base. We assume that the parallelogram S_i is "long", i.e.: (H) The width e_i of S_i is "small" w.r.t. $f_i = |ord(b_i) - ord(a_i)|$.
We have: $e_i/f_i < 1/20$. We consider a point $x^0 = (x_1^0, x_2^0) \in S$ (i.e., $x_1^0 \in S_1$ and $x_2^0 \in S_2$), and consider the following procedure $PROC0(x^0)$: Show that, if $x(0) = x^0$, then there exists $t \in [kT, (k+1)T)$: $x(t) \in S$ (i.e., $(x_1(t), x_2(t)) \in S_1 \times S_2$) (*recurrence* of S), and at t, the two components $x_1(t)$ and $x_2(t)$ of $x(t)$ are in phase, i.e.: $|\phi(x_1(t)) - \phi(x_2(t))| < \epsilon$ (*synchronization*).

The notion of *phase* $\phi(x_i(s))$, for $i = 1, 2$ of component $x_i(s)$ at time s, remains to be defined in this framework. From a general point of view, one can suppose that, during its traversal of S_i, the phase of the point $x_i(s)$ varies, after normalization, between 0 and 1. As S_i is of small dimension with respect to the orbit of the subsystem i, we can assimilate the trajectory described by $x_i(s)$ in S_i to a straight line segment whose ordinate varies from $ord(a_i)$ to $ord(b_i)$.

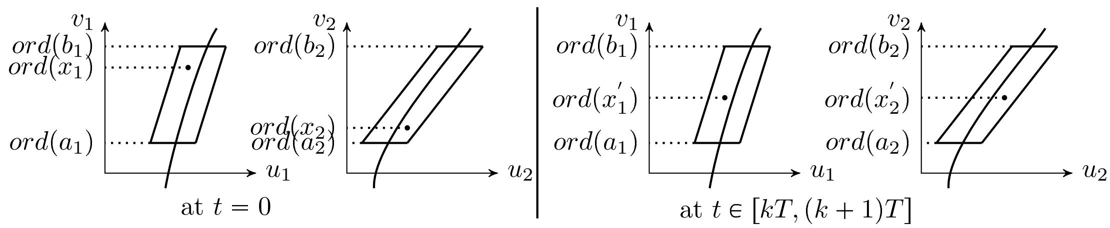

Figure 1: Scheme of S_1 and S_2 at $t = 0$ (left) and for some $t \in [kT, (k+1)T)$ (right).

Moreover, we can assume that on this small fragment of orbit, the *phase velocity* is *constant*. Given a point of $x_i(s)$ of $S_i \equiv (a_i, b_i, e_i)$ at time s, we can thus define its *phase* (in a "linearized" and "normalized" manner w.r.t. S_i) by: $\phi[x_i(s)] = (ord(x_i(s)) - ord(a_i))/(ord(b_i) - ord(a_i))$, where $ord(x_i(s))$ denotes the ordinate of $x_i(s)$. See Fig. 1.

2 Symbolic Reachability using Euler's method

The above procedure $PROC0$ takes a *point* of S as input. So it is not possible to prove the synchronization of *all* the points starting at S, since they are in infinite number. We thus need to consider a *symbolic* version of $PROC0$ which takes a *dense subset of points* as input. Such subsets are considered here under the form of "(double) ball" of the form $B = B_1 \times B_2$, where $B_i \subset \mathbb{R}^m$ $(i = 1, 2)$ is a ball of the form $\mathcal{B}(c_i, r)$ with $c_i \in \mathbb{R}^m$ (*centre*) and $r \in \mathbb{R}_+$ (*radius*).

Let $B^0 = \mathcal{B}(c_1^0, r^0) \times \mathcal{B}(c_2^0, r^0) \subset \mathbb{R}^m \times \mathbb{R}^m$, with $c_i^0 \in \mathbb{R}^m$ $(i = 1, 2)$ and r^0 positive real. As a symbolic method, we use here the *symbolic Euler's method* [LCDVCF17, Fri17] in order to compute (an overapproximation of) the set of solutions starting at B^0.

We define for $t \geqslant 0$: $B^{euler}(t) = \mathcal{B}(c_1(t), r(t)) \times \mathcal{B}(c_2(t), r(t))$, where $(c_1(t), c_2(t)) \in \mathbb{R}^m \times \mathbb{R}^m$ is the approximated value of solution $x(t)$ of $\dot{x} = f(x)$ with initial condition $x(0) = (c_1^0, c_2^0)$ given by *Euler's explicit method*, and $r(t) \approx r^0 e^{\lambda t}$ is the *expanded* radius using the *one-sided Lipschitz constant* λ [Söd06, AS12] associated to f (see [Fri17] for details) [1]. It is shown in [LCDVCF17] that $B^{euler}(t)$ contains all the solutions $x(t)$ that start at B^0:

$B^{euler}(t) \supseteq \{x(t) \mid x(0) \in B^0\} \equiv \{(x_1(t), x_2(t)) \mid (x_1(0), x_2(0)) \in \mathcal{B}(c_1^0, r^0) \times \mathcal{B}(c_2^0, r^0)\} (*)$

Given a ball $B = B_1 \times B_2 \subset \mathbb{R}^m \times \mathbb{R}^m$, the symbolic version of $PROC0$ is defined as follows: Let $B^0 := B$. Show that there exists $t \in [kT, (k+1)T)$:

1'. $B^{euler}(t) \subset S$, i.e.: $\mathcal{B}(c_i(t), r(t)) \subset S_i$ for $i = 1, 2$. (*recurrence*)

2'. $|phase(c_1(t)) - phase(c_2(t))| \leqslant \epsilon$ (*synchronization*)

Note that, since $\mathcal{B}(c_i(t), r(t)) \subset S_i$ $(i = 1, 2)$ by (1'.), we have: $r(t) \leqslant 0.5 * \min(e_1, e_2)$ (**)

Given S_i $(i = 1, 2)$ defined as a parallelogram (a_i, b_i, e_i), in order to show the phenomenon of phase synchronization, we first *cover* S_i with a *finite* set $\{B_{j,i}\}_{j \in J_i}$ of balls $B_{j,i} \subset \mathbb{R}^m$ (i.e., for $i = 1, 2$, $S_i \subset \bigcup_{j \in J_i} B_{j,i}$). From (1'.), (2'.), (*) and (**) , it follows:

Proposition: Given a covering $\{B_j\}_{j \in J_i}$ of S_i $(i = 1, 2)$, if, for all $(j_1, j_2) \in J_1 \times J_2$, $PROC1(B_{j_1} \times B_{j_2})$ succeeds, then, for all initial condition $(x_1^0, x_2^0) \in S$, there exists $t \in [kT, (k+1)T)$ such that $(x_1(t), x_2(t)) \in S$ with $|phase(x_1(t)) - phase(x_2(t))| \leqslant \epsilon + \min(e_1/f_1, e_2/f_2)$

[1] The value of λ is defined "locally", and varies according to the position of $x(t)$. For regions where $\lambda < 0$, the value of $r(t)$ is constant; the value of $r(t)$ increases when $x(t)$ occupies a region where $\lambda > 0$.

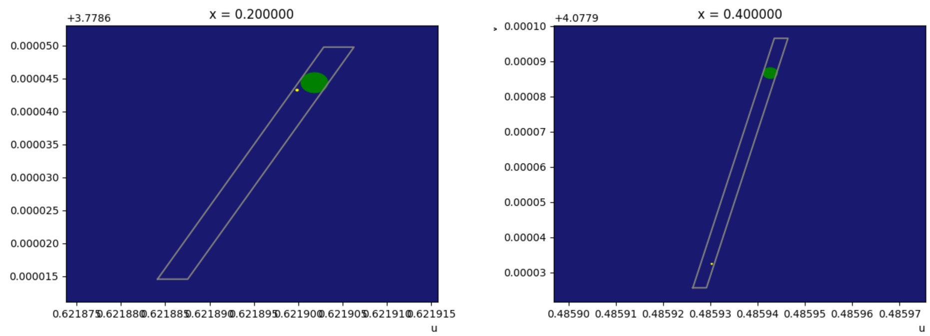

Figure 2: Synchronization of a ball, located initially (yellow), after 1 cycle (green).

When $\epsilon \ll \min(e_1/f_1, e_2/f_2)$, the final difference of phase between $x_1(t)$ and $x_2(t)$ is practically upper bounded by $\min(e_1/f_1, e_2/f_2)$. Since, by (H), e_i is "small" w.r.t. f_i, we know by Proposition that, if $PROC1$ succeeds for a set of balls covering S, then: For any initial point $(x_1^0, x_2^0) \in S$, there exists $t \in [kT, (k+1)T)$ such that $x_1(t)$ and $x_2(t)$ are *almost in phase*. In particular, even if $|phase(x_1^0) - phase(x_2^0)| \approx 1$, we have: $|phase(x_1(t)) - phase(x_2(t))| \approx 0$.

3 Example: Brusselator Reaction-Diffusion

We consider the 1D Brusselator partial differential equation (PDE), as given in [CP93]. Here we consider a state of the form $x(y,t) = (u(y,t), v(y,t))$ where $y \in \Omega = [0, \ell]$ is the spatial location. The PDE is of the form:

$$\begin{cases} \frac{\partial u}{\partial t} = A + u^2 v - (B+1)u + \sigma \nabla^2 u \\ \frac{\partial v}{\partial t} = Bu - u^2 v + \sigma \nabla^2 v \end{cases} \tag{1}$$

with boundary condition: $u(0,t) = u(\ell,t) = 1$, $v(0,t) = v(\ell,t) = 3$,
and initial condition $x_0(y) = (u(y,0), v(y,0))$ with: $u(y,0) = 1 + sin(2\pi y)$, $v(y,0) = 3$.

Let: $A = 1, B = 3, \sigma = 1/40$, $\ell = 1$. We transform the PDE into a system of ODEs by spatial discretization using a grid of $N+1$ points with $N = 4$ (i.e.: $y_i = \frac{i\ell}{N+1} = 0.2i$ for $i = 1,2,3,4$). We thus consider that we have 4 oscillators of state $x(y_i,t) = (u(y_i,t), v(y_i,t))$ with initial conditions $x(y_i,0) = (u(y_i,0), v(y_i,0))$ $(i = 1,2,3,4)$. These oscillators are coupled by a Laplacian matrix accounting for the continuous diffusion process;the size of the resulting global ODE is $N \times n = 4 \times 2 = 8$. By using symmetry, we can reduce the problem to 2 plans only $(x_1(t) = x_4(t) \wedge x_2(t) = x_3(t))$. $\tau = 2 * 10^{-4}$, period $T = 34564 * \tau$, time $t = 5T + u$, expansion factor after one cycle (5 periods) $= 1.8^5 = 30$.

In Fig. 2, we have depicted an initial ball (yellow) with a center of coordinate $(0.6219, 3.778643)$ in the plan $x = 0.2$ and $(0.485930, 4.077932)$ in the plan $x = 0.4$. its radius is $3.5 * 10^{-8}$. After 1 cycle, the image of the yellow ball is the green ball of center $(0.621902, 3.778644)$ in the plan $x = 0.2$ and $(0.48594267, 4.077987)$ in the plan $x = 0.4$; the radius is now $1.512 * 10^{-6}$. The phase of the initial ball center is 0.823 in the plan $x = 0.2$ and 0.087 in the plan $x = 0.4$, so the difference of phase between the 2 plans is 0.736. The phase of the green image ball is 0.874614 in the plan $x = 0.2$ and 0.875 in the plan $x = 0.4$, so the difference of phase is $1.23 * 10^{-5}$.

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