Dynamic Network Congestion Games

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1 Context

Congestion Games are classical type of games studied in the game theory, in which each player chooses resources, and pays cost depending on how many players have chosen it. In network congestion games (NCGs, for short), players choose simple path from their source to target - by a simultaneous and single-shot choice at the beginning of the game. We introduce a variant of these network games - where players choose their paths dynamically, step by step, observing other players and their own history - hence the name *dynamic* NCGs. Moreover, the cost of each player is affected by other players only if they take the same transition simultaneously- which is also different from most of the traditional models of network games.

Traditionally, in the context of selfish behaviors, most of the studies involve providing bounds for price of anarchy (PoA) and price of stability (PoA) for a class of games. We study the impact of selfish behaviors from a complexity perspective. Our study involves standard concepts of game theory in case of dynamic NCGs: Social optima, Nash Equilibria, Subgame Perfect Equilibria, Price of Stability and Price of Anarchy. Our contributions are following. The existence of a strategy profile with social cost bounded by a constant is in PSPACE and NP-hard. Nash Equilibria always exist in dynamic NCGs. Computing Nash equilibria with bounded cost can be done in doubly-exponential time, and the associated decision problem is in EXPSPACE. Finally, we can compute price of anarchy and price of stability in doubly-exponential time. We also prove the decidability of the bounded SPE problem.

2 Contributions

Preliminaries. A dynamic network congestion game (dynamic NCG, for short) is a pair $\mathcal{G} = \langle \mathcal{A}, n \rangle$, where $\mathcal{A}$ is the game arena, which is a finite directed graph with edge labels and common source-target vertex pair for all players, and n is the number of players. Edges in the arena are labeled by non-decreasing latency functions from $\mathbb{N}$ to $\mathbb{N}$ that are piecewise-affine. The game proceeds in rounds. Starting with the source vertex **src**, in each round, each player chooses an edge simultaneously from its current vertex, with the aim of reaching the target vertex **tgt** with minimal accumulated cost. A play ρ in the game is a sequence of triplets of the form $((c_i, \bar{e}_i, c_{i+1}))_{i \in \mathbb{N} \cup \{0\}}$, where each c_i is the configuration at i^{th} round - a tuple depicting positions of n players with $c_0 = (src)_{i \in [n]}$. We assume the target vertex **tgt** to be co-reachable from each vertex, which ensures that each play in our game is finite. A history h is a finite prefix of a play in the game.

A player's cost for using an edge depends on the number of players taking that edge simultaneously, and also the latency function by which the edge is labeled. A player's cost in a play ρ is the sum of cost incurred by all the edges that the player has used.

A strategy of Player i is a map σ_i that assigns an edge to a history such that the edge can be taken by Player i from the final configuration of that history. The one-shot, simultaneous strategies, that are effectively paths from source to target, are called blind strategies in our setting. We use blind strategies to achieve some results in the more general case.

Standard game theory concepts. Here let us recall some standard game theory concepts that we will be talking about. A strategy profile is called *NE profile* if for any player there does not exist any beneficial single-player deviation from any round of the game. An strategy profile is a blind NE if it is an NE profile when the strategy space is restricted to the set of blind strategies. PoA (resp. PoS) is the ratio between the social cost of best (resp. worst) NE with that of the social optimal profile.

Now for a strategy profile σ, let us denote by σ_h the strategy profile such that $\sigma_h(h') = \sigma(hh')$. A strategy profile σ is called an SPE if for every history h the profile σ_h is an NE profile. Alternatively, we can say an SPE is an NE profile at every sub-game.

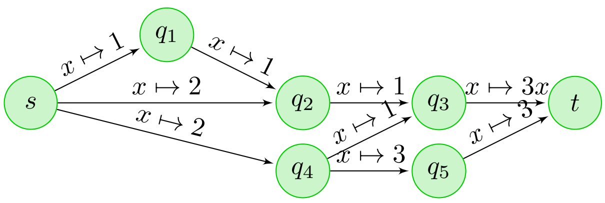

Figure 1: Example of a network congestion game where dynamism matters.

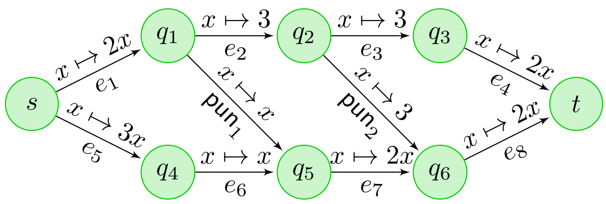

Figure 2: An arena on which blind Nash equilibria are sub-optimal.

Related works. Having defined the basic setting of dynamic NCGs, here we emphasize how we differ from the classical model of network congestion games:

- In our setting, players choose their paths dynamically, one edge at each round, depending on the history of the game.

  A dynamic version of resource allocation games (DRAG, in short) has been studied in [2] where every player allocates one resource at each round in order to fulfill their job which is allocating one set from their sets of target set of resources. While resources from *ordered* DRAGs (where players allocate resources in some given order) seem to correspond to edges, and sets of resources to paths in dynamic NCGs, the model only covers acyclic arena of dynamic NCG. Hence positive results from DRAG can't be carry forward to dynamic NCGs. On top of that, they include empty order in ordered DRAGs which prevents even negative results to carry over in dynamic NCGs. Hence we study our model separately.

- In our settings, a player's cost does not get affected by other players if they don't take the edge simultaneously at the same time.

  Traditionally, it has not been the case. For example consider the well studied routing games from [8, 9, 5] or even the dynamic RAG from [2], where POA and PoS has been studied in the context of congestion. In all of those, players bear congestion effects in their cost even if they use the resources at different times (for some like DRAG[2] the strategies are of the form sets, so no notion of time can be materialized a priori). On the other hand, in [1], the time factor has been incorporated in a more rigorous way, but they do not consider a dynamic strategy space. In that model, each player chooses a timed path (like blind strategy in our case except there is a time component which tells when the player will take which edge), *waits* on the nodes where congestion cost incurs on each player depending on how many players wait together and for how much time. Our model somewhat lies in between, and thus we expect more positive results than some more rigorous ones while maintaining a different (synchronous) take on the congestion effect.

Example. We illustrate the differences between classical NCGs and dynamic NCGs on the example from Figure 1. In this game, players want to move from source s to the target vertex t. The costs of all transitions are constant (thus independent on the number of players using them), except for the transition from q_3 to t which has cost $x \mapsto 3x$, thus proportional to the number of players using it. To illustrate the difference between NCGs and dynamic NCGs in the cost computation, consider two players choosing the following paths: $\pi_1 = s \to q_1 \to q_2 \to q_3 \to t$ and $\pi_2 = s \to q_4 \to q_3 \to t$. In the standard setting of [7], the cost of each player is 9, because they suffer congestion on the last transition. By contrast, in the dynamic setting, the two players do not use the transition $q_3 \to t$ simultaneously, so their cost is only 6.

On this example, one can also see that dynamically selecting paths may change the costs players incur. Consider again two players, Player 1 starting with the transition $s \to q_4$, and Player 2 with one of the other two transitions from s. In case Player 2 starts with the upper transition $s \to q_1$, then it is best for Player 1 to choose the path $s \to q_4 \to q_3 \to t$: its cost is then 6, instead of 8 if they would go via q_5. If Player 2 starts with $s \to q_2$, then Player 1 should rather choose $s \to q_4 \to q_5 \to t$: their cost is then 8 instead of 9. We see here that Player 1 benefits from adapting their strategy after the first move, observing the choice of Player 2.

Our contributions with brief ideas. We take a computational-complexity viewpoint to study dynamic network games. We first establish the complexity of computing the social optimum, which we show to be in PSPACE and NP-hard. We encode the configurations into an abstract weighted configuration graph, by only considering Parikh image of the configurations, and the minimum weight path in that graph gives us the social optimum in PSPACE. Note that, the number of players, n, can be encoded in binary, hence the standard configuration graph is doubly-exponential, and that is why we need to abstract it out for obtaining a lower complexity. NP-hardnes is shown by a reduction from the partition problem, adapted from [6].

Next we show that Nash Equilibria exist in dynamic NCGs. First we establish existence of blind Nash Equilibria using potential games. We use the best-response problem, which is polynomial in our setting, and then observe the convergence of best-response dynamics by providing a potential function, similar to [7]. The best-response dynamics converges in case of blind strategy profiles, but we could not show the same for general strategy space (dynamic ones). But we are able to show that blind NE profiles are indeed general NEs. Because if we consider a blind strategy profile and allow general strategy single-player deviation, we can see that any general strategy deviation can be simulated by a blind strategy deviation hence proving our claim. This finally establishes the existence of Nash Equilibria.

We can show the existence of NCG with a Nash Equilibrium π such that for all blind Nash equilibria π', we have $cost(\pi) < cost(\pi')$. Figure 2 shows an instance of this phenomenon. This strengthens our justification of considering dynamic NCG instead of normal NCGs.

Then we study two constraint NE problems where, given a dynamic NCG and a number $M \in \mathbb{N}$, we ask: (1) ($\exists$-NE) whether there exists a NE profile the social cost of which is less than M, (2) ($\forall$-NE) whether the social cost of all NE profiles is less than M.

We first characterize the outcomes of NE profiles, and obtain necessary constraints for a path in the configuration graph to be the outcome of a NE profile.

Then we consider a graph in which the vertices are the configurations augmented with an additional n-tuple, where the values come from $\mathbb{N} \cup \{\infty\}$, and where the edges are precisely the transitions in the configuration graph, weighted by the social cost of the transition. We non-deterministically guess a path (depicting a play in the original game) in this graph, and verify whether that path satisfies the constraints given by the aforementioned characterization, and the additional condition given by the bound M. The augmented n-tuples keeps track of the constraints for the path, and those values along the path depend on the weights of the edges on that path.

For $\forall$-NE problem, we adjust the aforementioned graph by changing the signs of the weights of the edges. Using the fact there would not be a negative cycle in the graph (because the values in the n-tuples are decreasing), we can verify whether there exists a path satisfying the constraints like above, and maintaining the bound - which in turn, is equivalent to checking whether all path that satisfies the NE outcome constraints also have social cost less that M.

These two decision procedures are in EXPSPACE, and finally we can obtain the PoA and PoS in doubly exponential time.

Now, in dynamic setting we encounter non-credible threats in case of NE profile. To discard these, we consider another stability concept, better fitted for the dynamic setting, which is SPE. Now [2] shows that SPE might not exist in dynamic RAGs, but their argument doesn't work in our setting. On the other hand, the standard argument for showing existence of SPE in turn-based games, like in [3], also fails to work in the simultaneous setting. We are yet to show either of existence/failure of existence of SPE in case of dynamic NCGS. But we obtain an algorithm which can decide whether given dynamic NCG there exists an SPE at all or not.

Here also we first characterize SPE outcomes. Starting with all possible plays, we develop an iterative procedure which discards plays that fail to satisfy the constraints developed from the characterization. These constraints also get updated in each round of iteration, resulting in discarding new set of plays in the next round, and finally producing only the set of SPE outcomes at the fixpoint of the iteration. This procedure is somewhat inspired from the same problem studied in the context of turn-based game by Thomas et al [4], and adapted to our simultaneous setting.

3 Future Directions

In conclusion, we introduced dynamic network congestion games, and studied the complexity of various decision and computation problems concerning social optima, Nash Equilibria and subgame perfect equilibria.

In the near future, we are looking forward to work on following directions: (1) Either proving existence of SPE for dynamic NCG, or having an example where SPE doesn't exist. (2) Having lower bounds on the aforementioned studied problems

A long term research objective is to establish how measures such as PoA, PoS, or costs of best Nash equilibria and social optima evolve with the number of players, seen as a parameter.

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