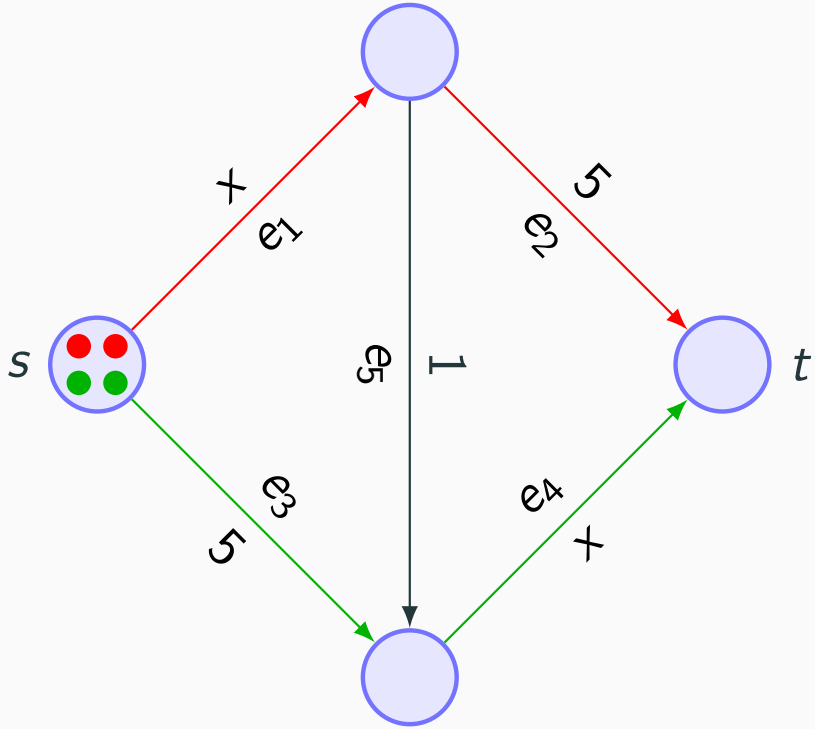
Dynamic Network Congestion Games

Joint work with Nathalie Bertrand, Nicolas Markey, and Ocan Sankur

Suman Sadhukhan, Inria Rennes

MOVEP, 23^{rd} June, 2020

Routing Games[1]



$$cost = 2 + 5 = 7$$

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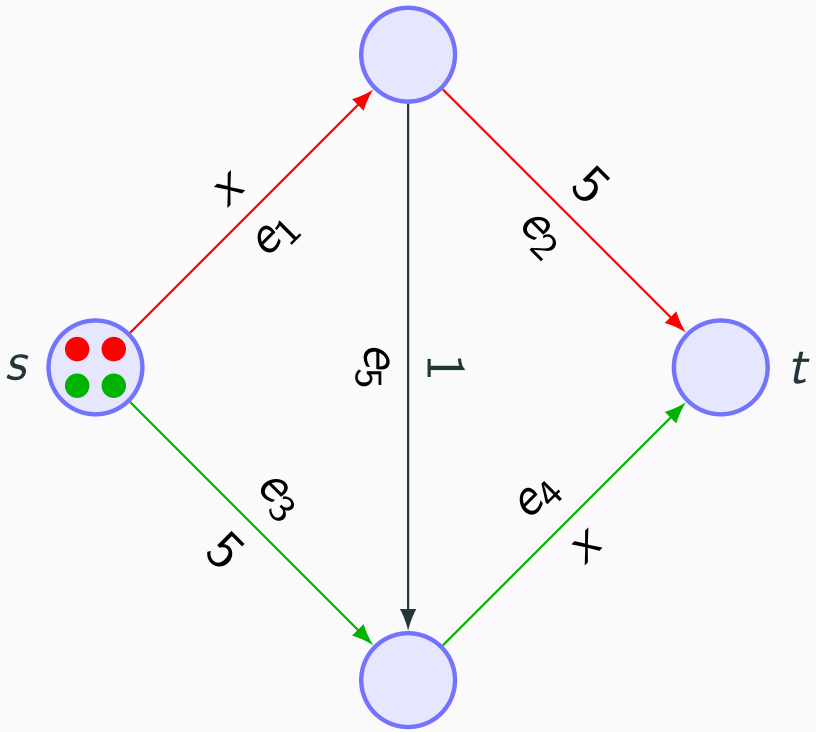
$$Total\ cost = 7 \times 2 + 7 \times 2 = 28$$

[1] Roughgarden 2005.

Routing Games[1]



"centralized"

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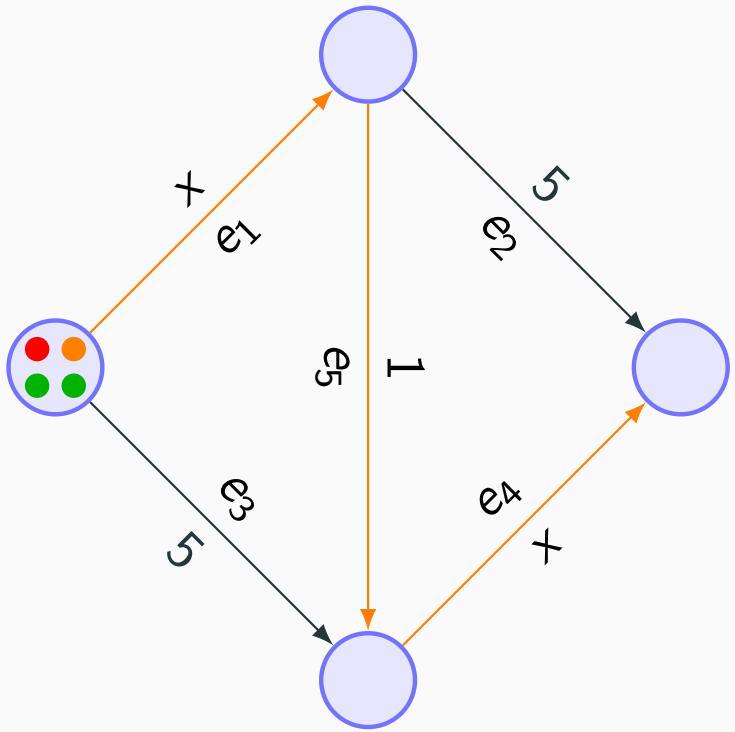


"selfish"

$cost = 7$

$cost = 5 + 3 = 8$

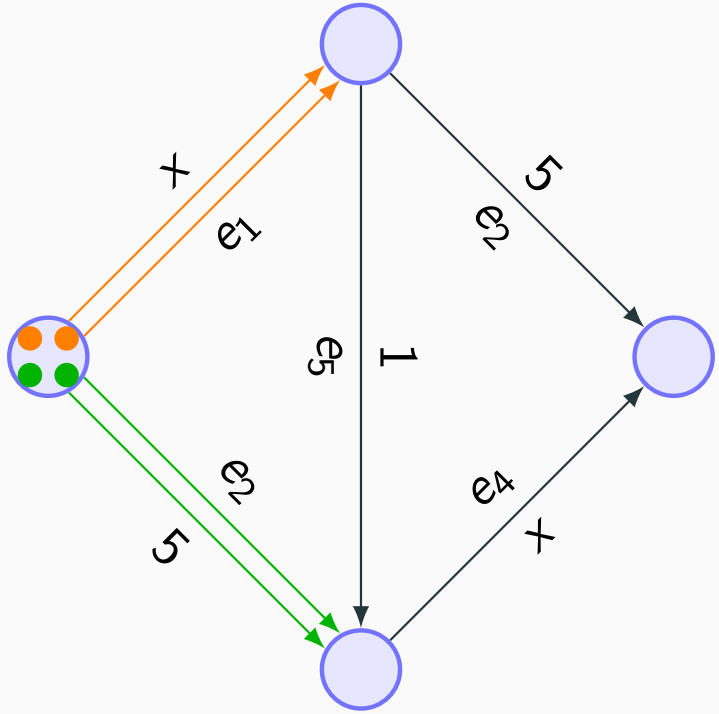
$cost = 2 + 1 + 3 = 6$

$Total\ cost = 7 + 8 \times 2 + 6 = 29$

[1] Roughgarden 2005.

Where we differ?

- Synchronicity: congestion cost only if players take an edge simultaneously
- Dynamic strategies: Strategies are not just paths from source to target



$cost = 2+$

$cost = 5+$

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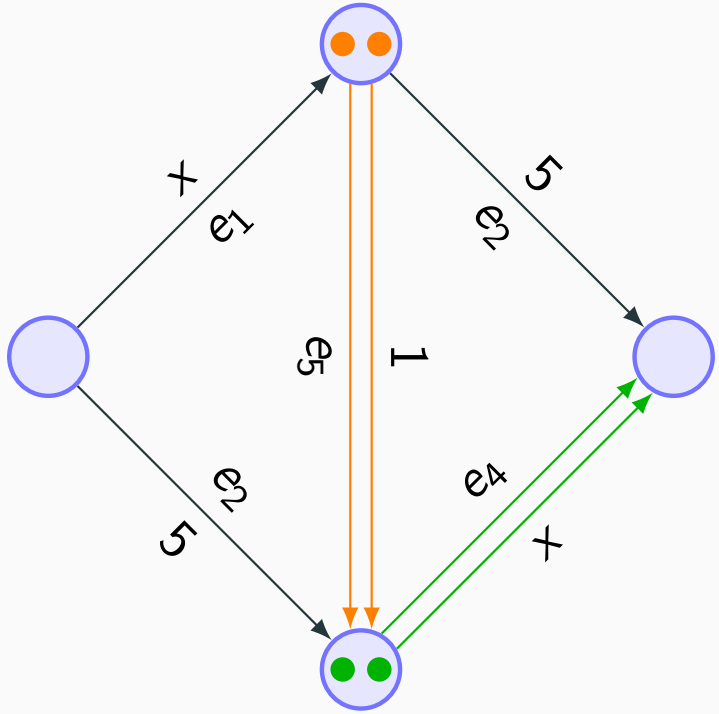

$cost = 2 + 1$

$cost = 5 + 2$

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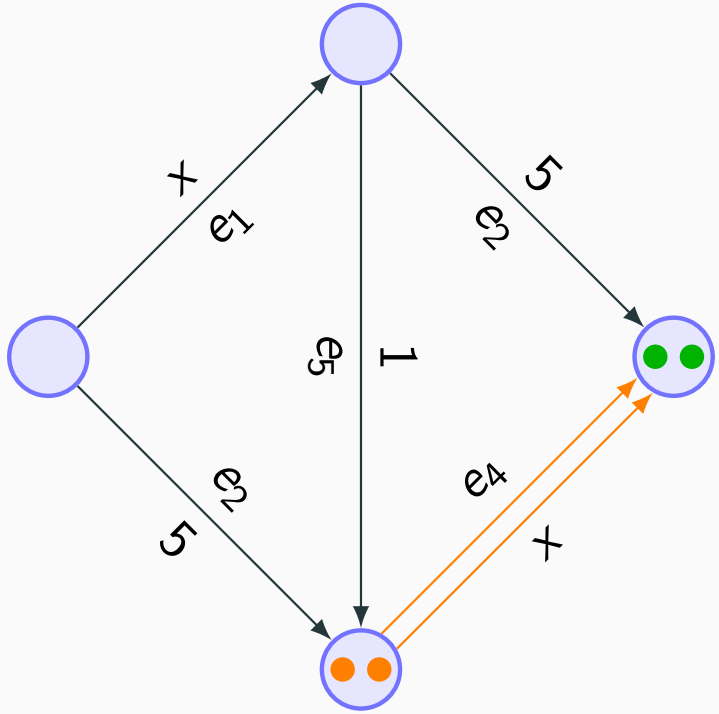

$cost = 2 + 1 + 2 = 5$

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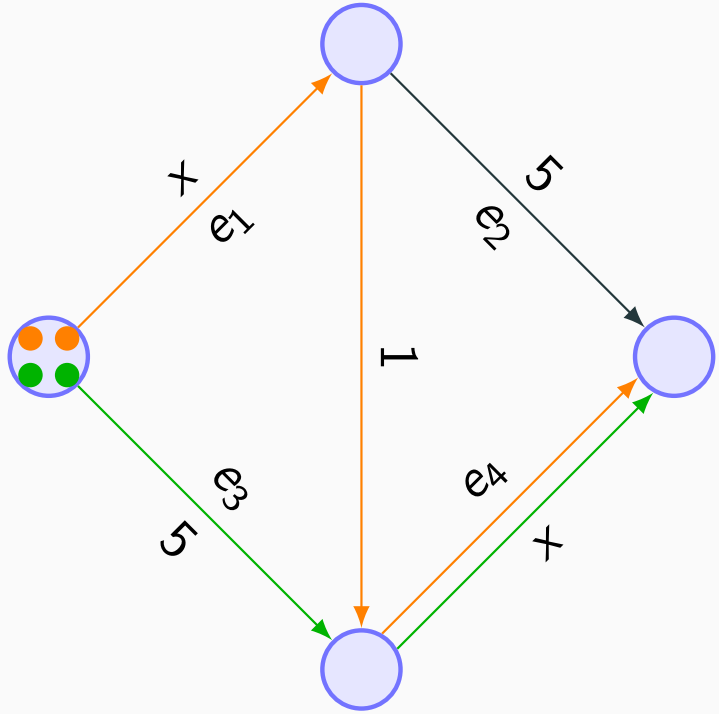

$cost = 2 + 1 + 2 = 5,\ old\ cost = 7$

$cost = 5 + 2 = 7,\ old\ cost = 9$

Where we differ?

- Synchronicity: congestion cost only if players take an edge simultaneously
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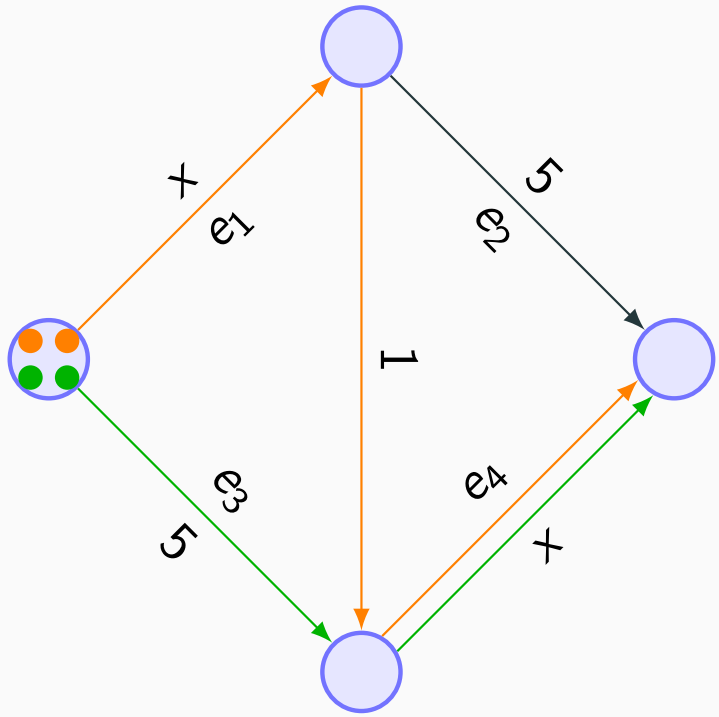
Related model: Dynamic Resource allocation games (DRAG)[2]

- Edges are resources, paths are sets of resources
- Ordered DRAG is acyclic: results do not carry forward to our model



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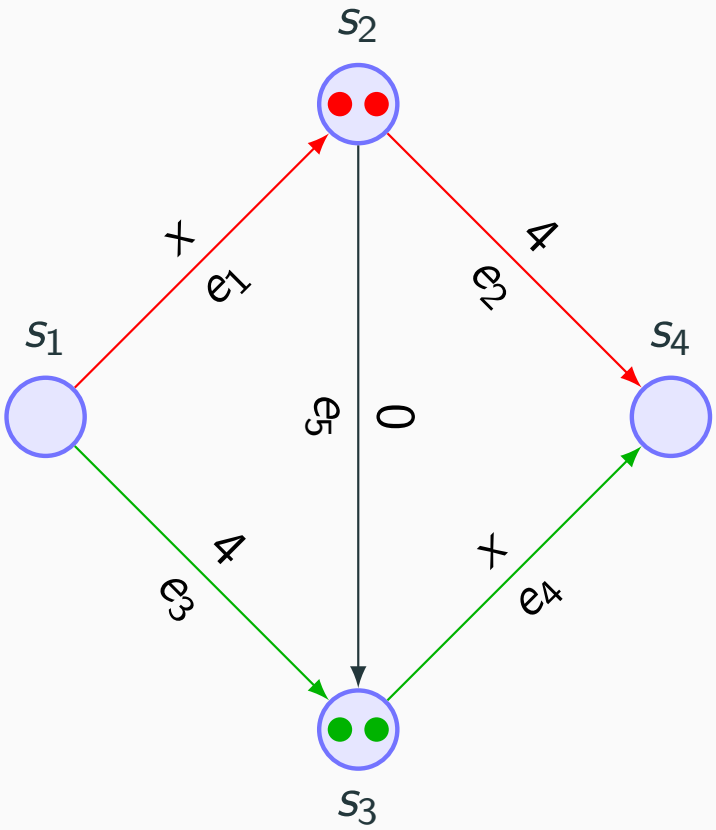
[2] Avni, Henzinger, and Kupferman 2016.

Our model: Dynamic Network Congestion Games



$\mathcal{G} = (V, E, \{cost_e\}_{e \in E}, k, s, t)$

$cost_e : \mathbb{N} \to \mathbb{N}$

k : Number of players

$s, t \in V$: source and target vertices

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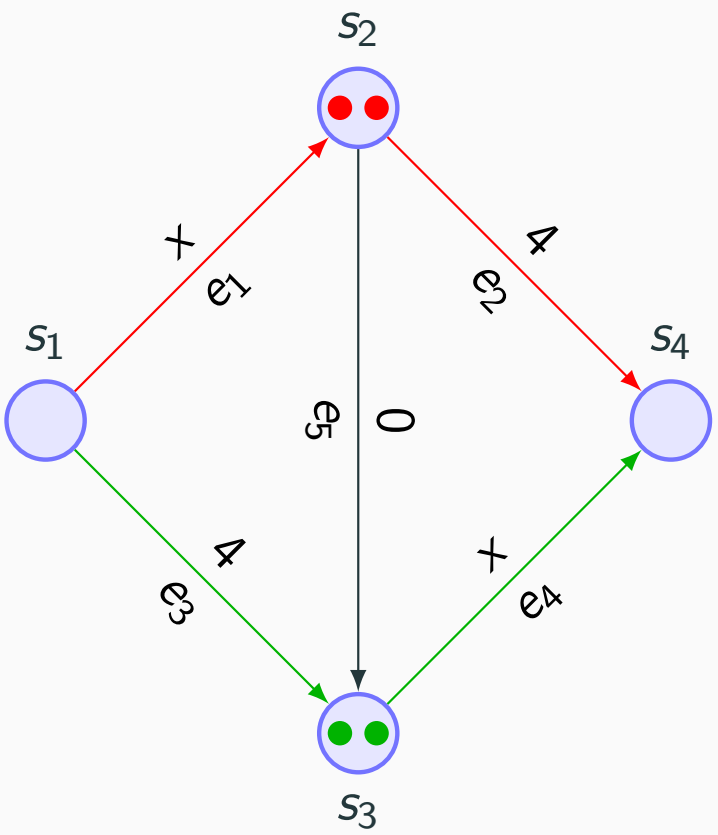

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Configuration $c = (s_2, s_2, s_3, s_3) \in \mathcal{C}$

$$cost_i(paths) = \sum_{e \in path_i} cost_e(.)$$

- Blind strategies : Players only observe history length, $\sigma_i^{bl} : \mathbb{N} \to E$.
- General strategies: $\sigma_i^{g} : \mathcal{C}^+ \to E$.

Standard Concepts

Nash Equilibrium (NE). A strategy profile $(\sigma_1, \sigma_2, \ldots \sigma_k)$ is a NE if no player has an incentive to deviate from its current strategy. $\rightsquigarrow$ "selfish"

Social Optimal (SO). A strategy profile $(\sigma_1, \sigma_2, \ldots \sigma_k)$ is a SO if $cost = \sum_i cost_i$ is minimum. $\rightsquigarrow$ "centralized"

Price of Anarchy (PoA)[a] $= \dfrac{\text{Total cost of "worst" NE}}{\text{Total cost of SO}}$

Price of Stability (PoS) $= \dfrac{\text{Total cost of "best" NE}}{\text{Total cost of SO}}$

[a] Koutsopias and Papadimitrou 2009.

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Questions:

- **Is PoA/PoS always well-defined?**
- **How to compute these measures?**

Is PoA/PoS always well-defined for our model?

Existence and Computation of NE: Blind Strategies

Best-Response Problem (BR). Given a strategy profile $\mathcal{P} = (\pi_1, \pi_2, \ldots \pi_k)$, can player i improve its cost by deviating? If so, what is its optimal strategy?

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- ▷ How to solve BR problem? Shortest weighted path problem
- ▷ Does this iterative procedure converge? Existence of Potential function[3]

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- ▷ How to solve BR problem? Shortest weighted path problem: PTIME
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Existence and computation of NE: General Strategies

Theorem
Blind strategy NE are general strategy NE.

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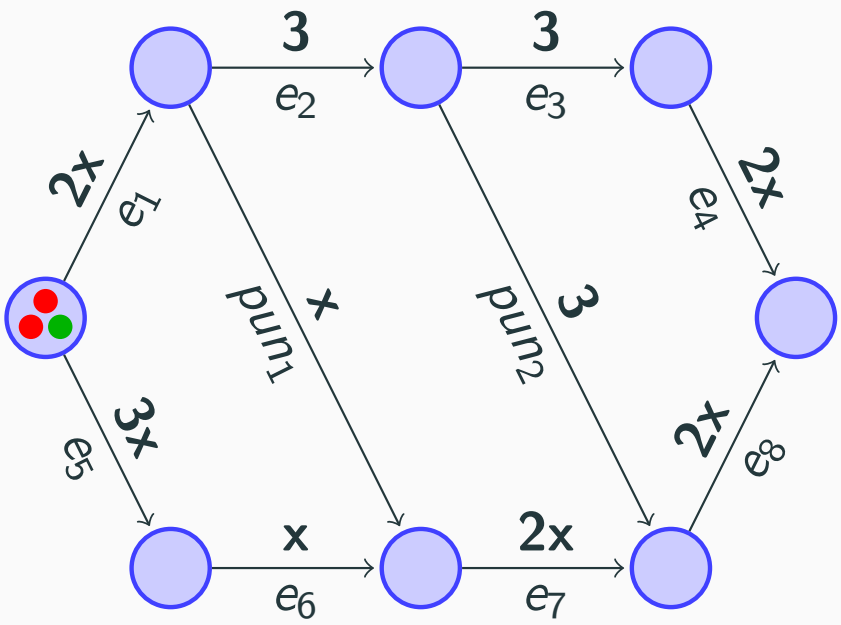

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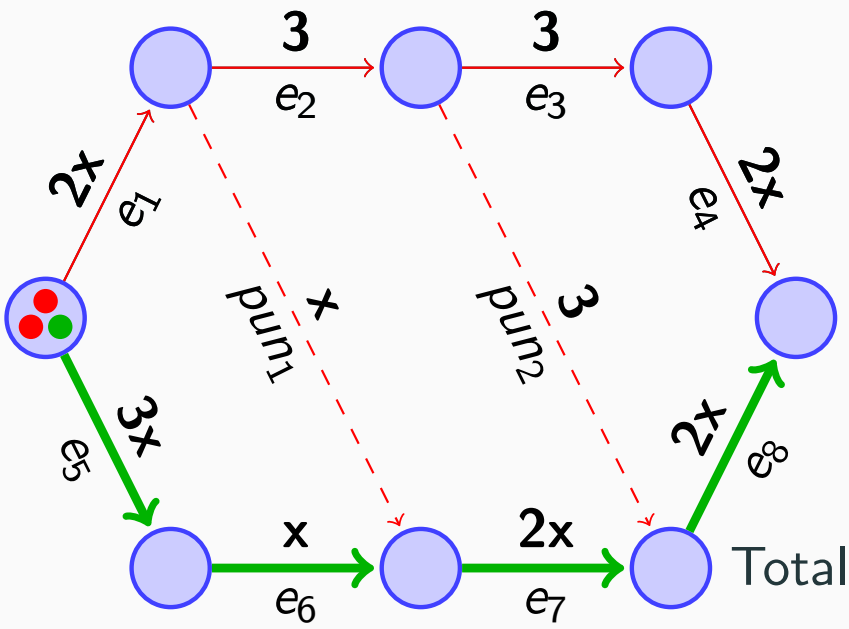

- cost = 14
- cost = 8

Total cost $= 14 \times 2 + 8 = 36$

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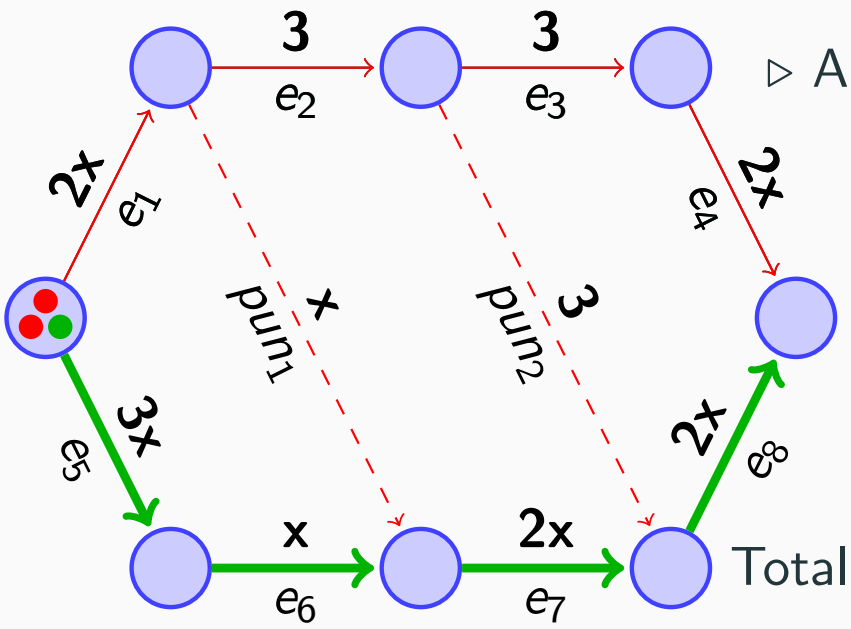


- ▷ All blind NEs have total cost at least 37.
- ▷ PoS depends on strategy space.

- cost $= 14$
- cost $= 8$

Total cost $= 14 \times 2 + 8 = 36$

How to compute PoS/PoA for our model?

Decision problem(s)

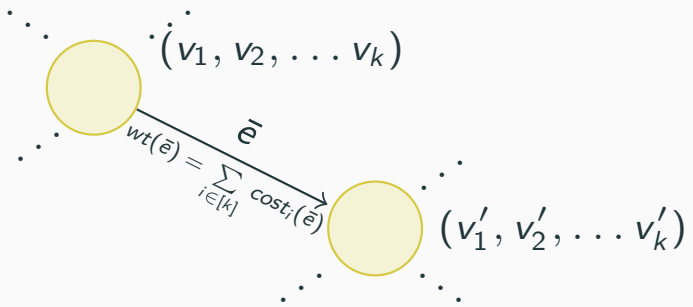
Social optimal: $\exists$ strategy profile.$\rightsquigarrow$ Computes SOpt
Input. A dynamic NCG $\mathcal{G}$, and $M_{opt} \in \mathbb{N} \cup \{0\}$.
Question. Does there exist a strategy profile with social cost $\leq M_{opt}$.

Best NE problem: $\exists$ NE profile.$\rightsquigarrow$ Computes PoS
Input. A dynamic NCG $\mathcal{G}$, and $M_{\exists} \in \mathbb{N} \cup \{0\}$.
Question. Does there exist a NE profile with social cost $\leq M_{\exists}$.

Worst NE problem: $\forall$ NE profile.$\rightsquigarrow$ Computes PoA
Input. A dynamic NCG $\mathcal{G}$, and $M_{\forall} \in \mathbb{N} \cup \{0\}$.
Output. Do all NE profiles in $\mathcal{G}$ have social cost $\leq M_{\forall}$.

Social Optimal Problem

Configuration Graph ($\mathcal{C}_\mathcal{G}$): $(V^k, \mathcal{E})$
Make it a weighted graph with weight function, $wt(\bar{e}) = \sum_{i \in [k]} cost_i(\bar{e})$



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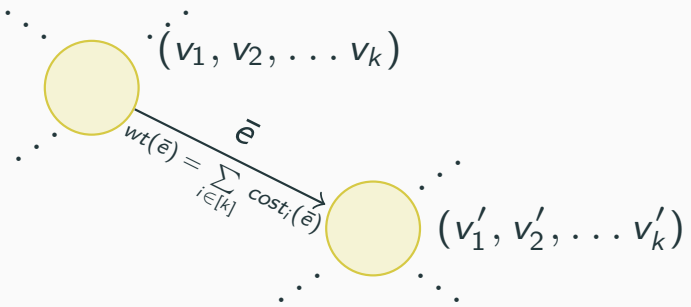

Shortest weighted path in this graph = the outcome of an SO
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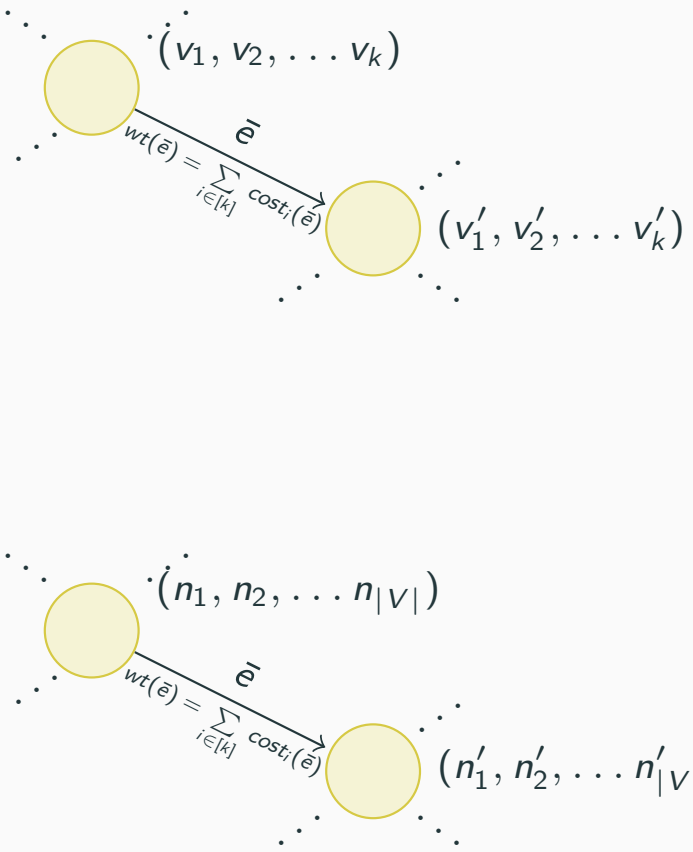
$\rightsquigarrow$ Configuration graph is double-exponential in k ☹



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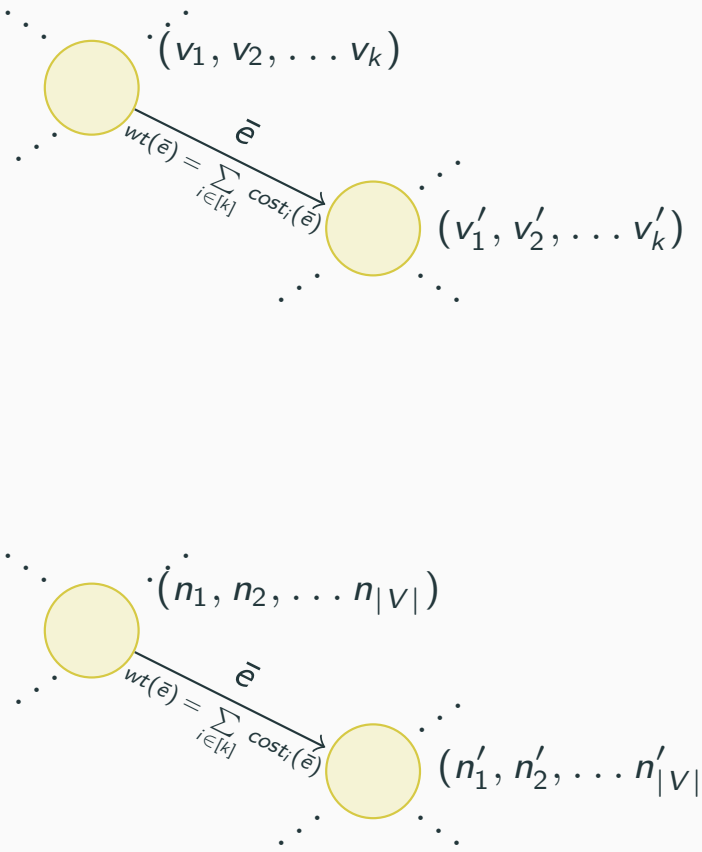
$\rightsquigarrow$ Can compute cost of SO from this! ☺
$\rightsquigarrow$ Configuration graph is double-exponential in k ☹

Configuration graph does not store any information about NE property
$\rightsquigarrow$ Can't compute cost of NEs from $\mathcal{C}_\mathcal{G}$ as it is!! ☹



PSPACE Complexity for $\exists$ strategy profile

Figure 1: An arena $\mathcal{A}$

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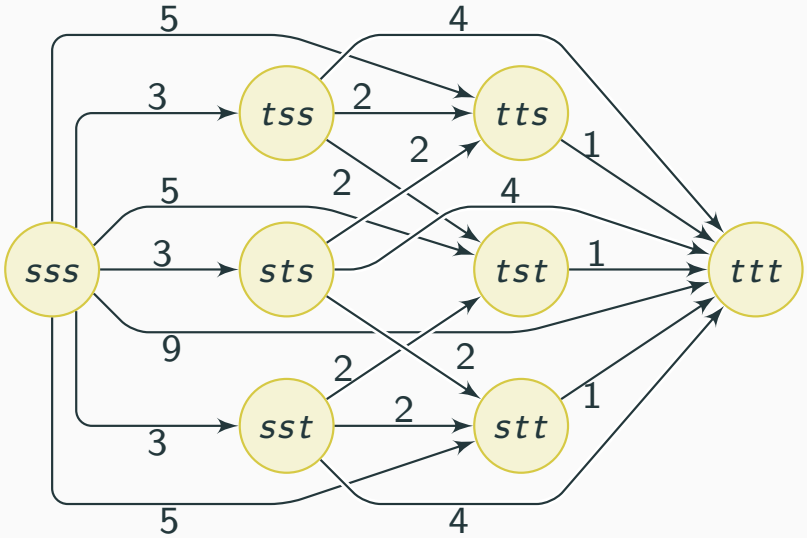


Figure 2: The configuration graph associated with $(\mathcal{A}, 3)$ (loops omitted)

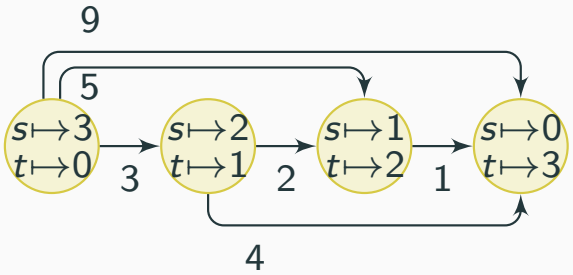


Figure 3: The abstract weighted graph associated with $(\mathcal{A}, 3)$ (loops omitted)

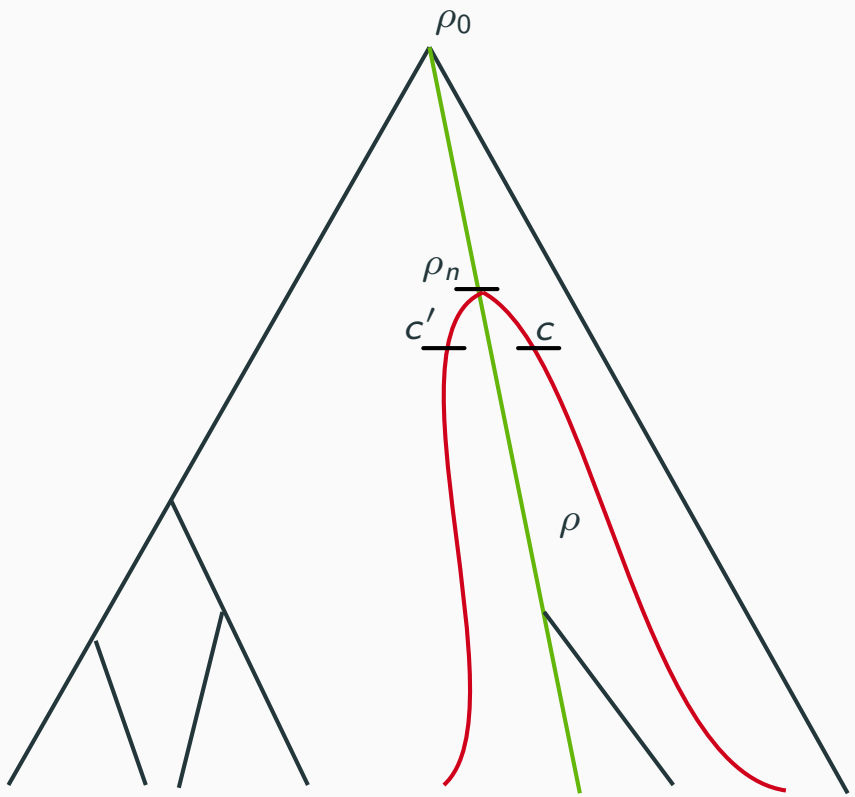
Characterization of outcomes of NE

Theorem
A play ρ is the outcome of an NE iff

$$\forall i \in [k], \forall n < |\rho|, \forall c \in Succ_\rho(i,n)$$
$$cost_i(\rho_{\geq n}) \leq val_{i,c} + cost_i(\rho_n, c)$$

where
$Succ_\rho(i,n)$ = siblings of ρ_{n+1} that can be reached from ρ by varying only player-i's choice
$=\{c | (\rho_n, c) \in \mathcal{E} \& c[j] = \rho_{n+1}[j] \forall j \neq i\}$

$val_{i,c}$ = cost that player-i can not avoid paying from configuration c if others form a coalition against that player
$= \min_{\sigma_i} \max_{\sigma_{-i}} cost_i^c((\sigma_i, \sigma_{-i}))$



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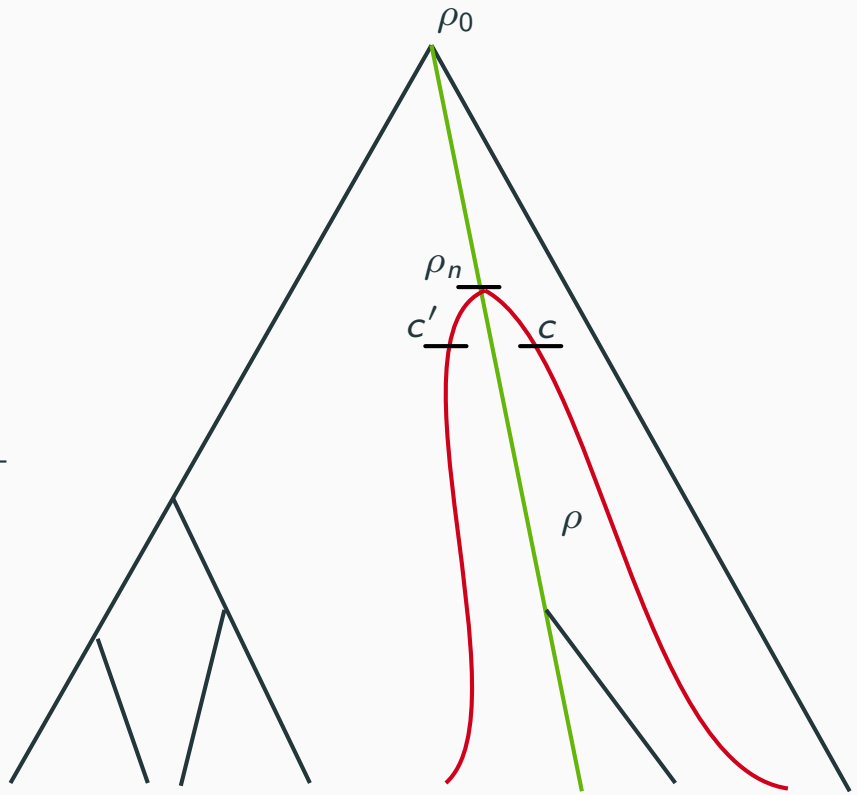
Computation of $val_{i,c}$ is reduced to the value computation of a 2-player turn-based game[a]

[a] Khachiyan et al. 2008.



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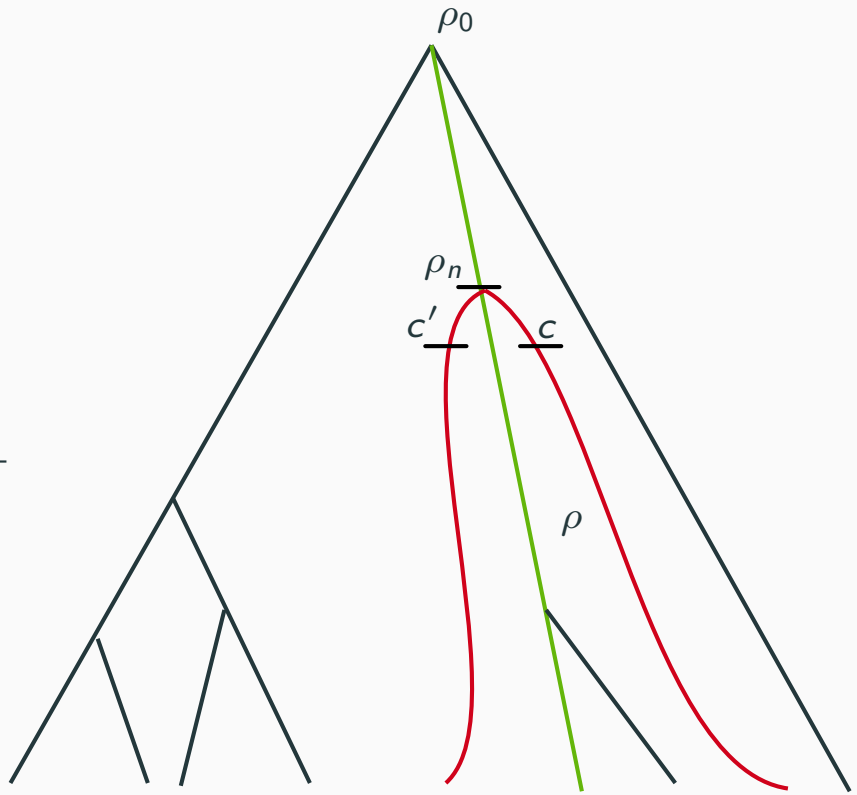

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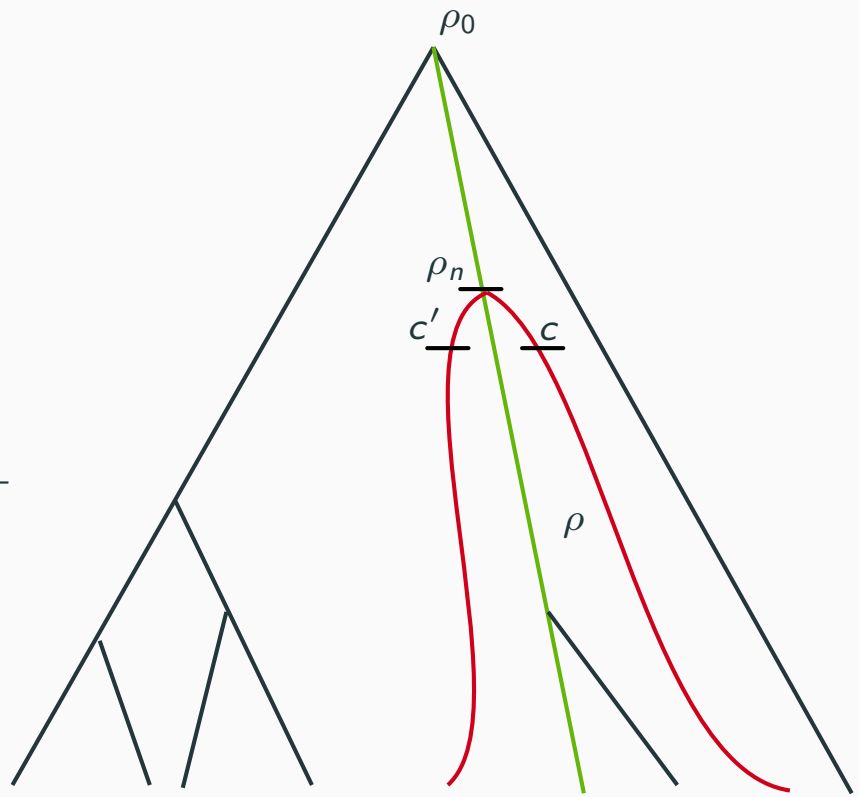
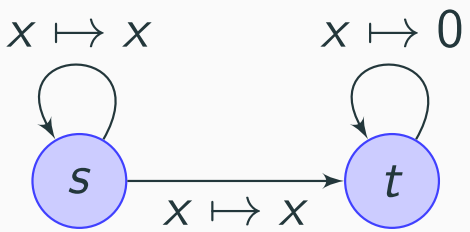


EXPSPACE algorithm for best & worst NE

$$m_i = \min\{\underbrace{m_i - cost_i(\bar{e})}_{\text{From history}}, \underbrace{\min_c \langle val_{i,c} + cost_i(\_, c) - cost_i(\bar{e})\rangle}_{\text{For current state}}\}$$

$\langle (s,s,s), (\infty,\infty,\infty)\rangle$



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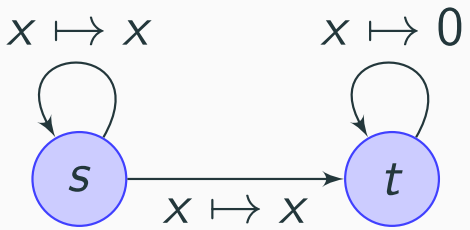

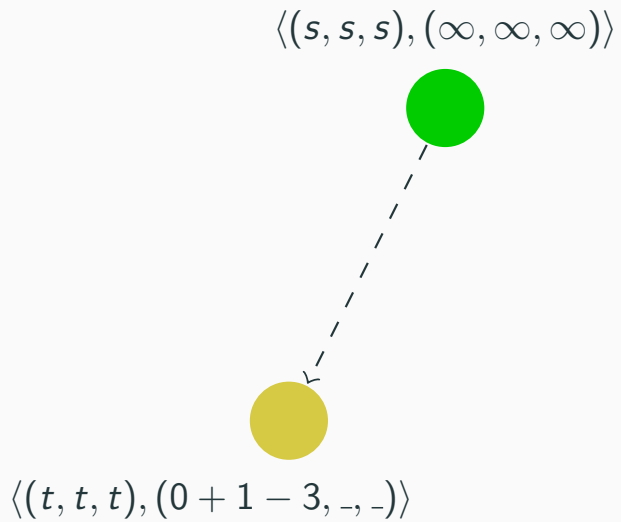

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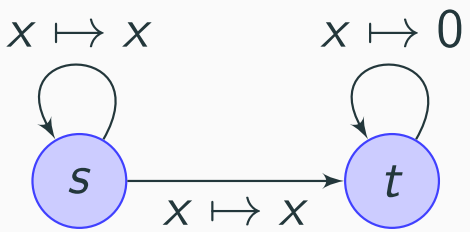

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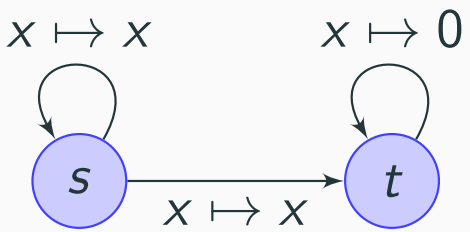

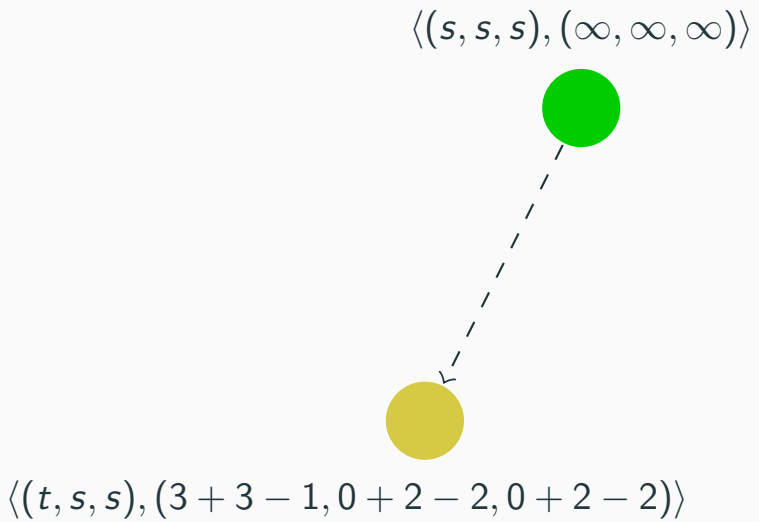

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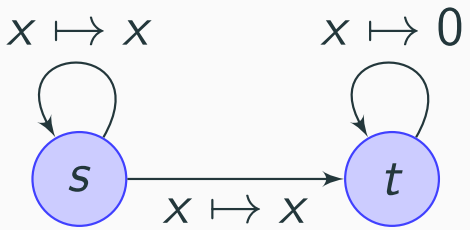

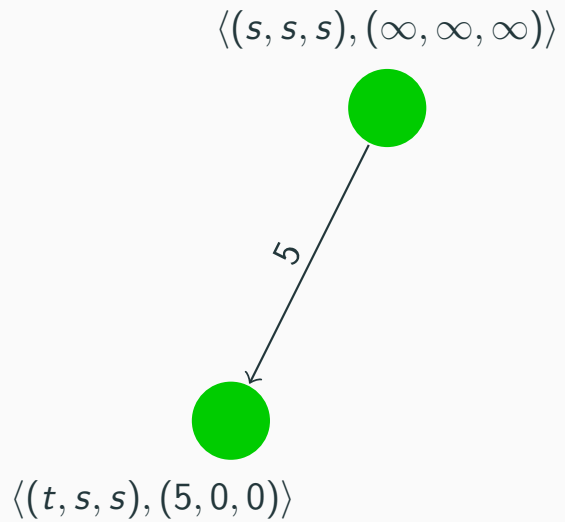

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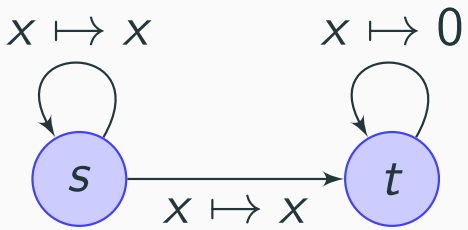

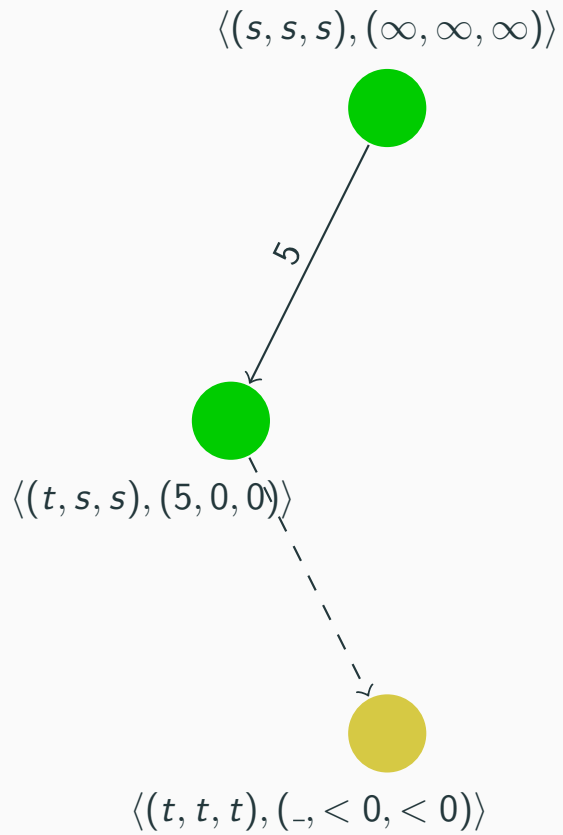

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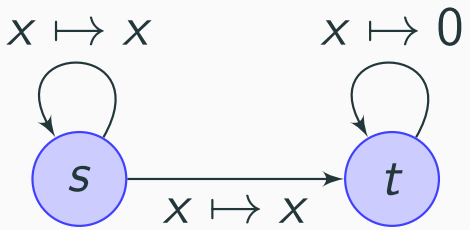

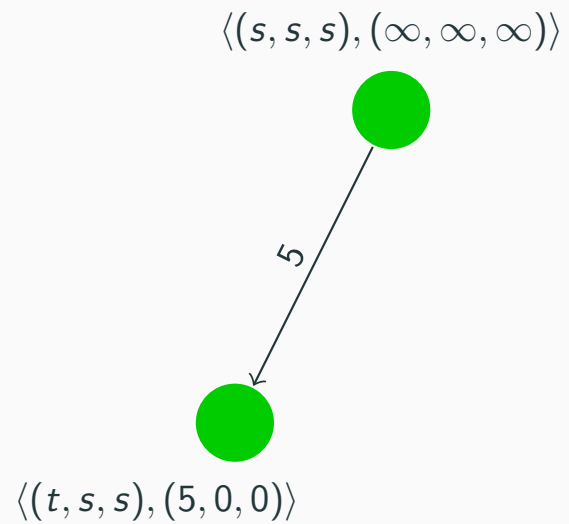

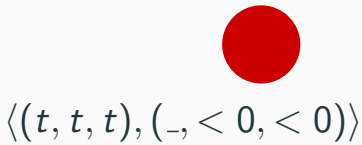

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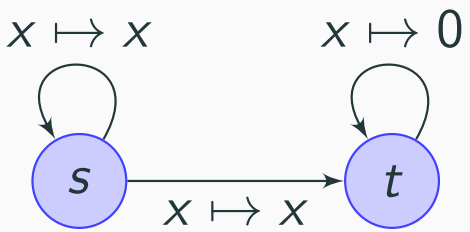
$\langle (s,s,s), (\infty,\infty,\infty)\rangle$



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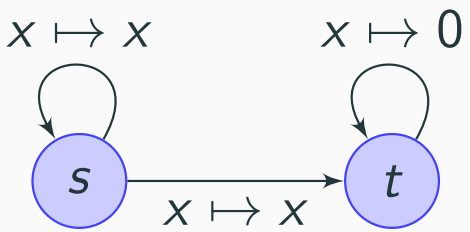

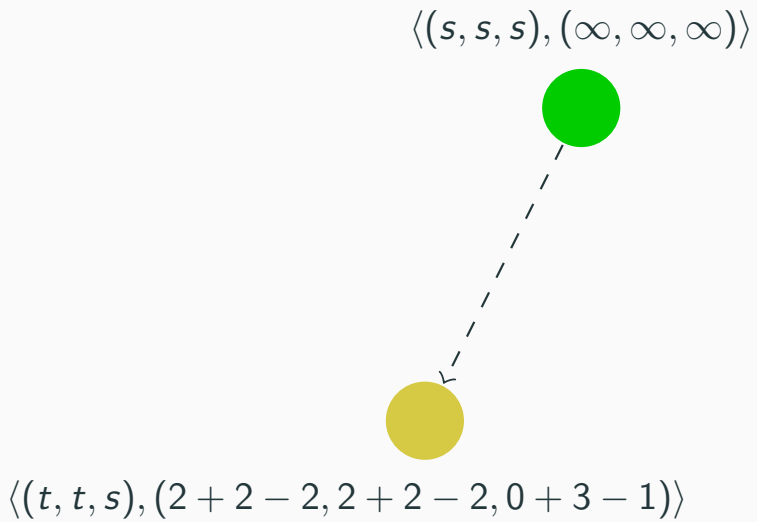

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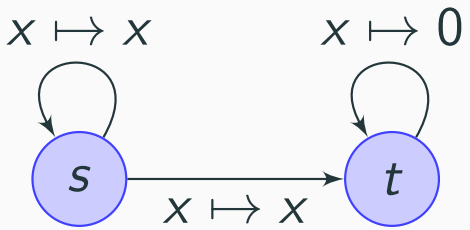

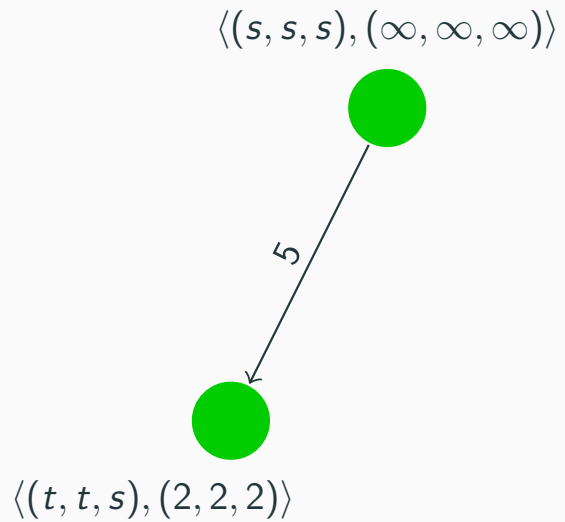

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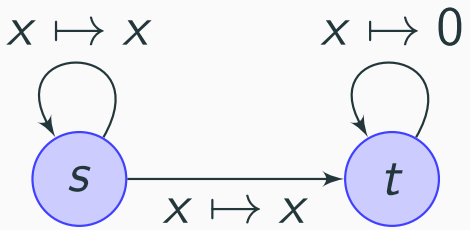

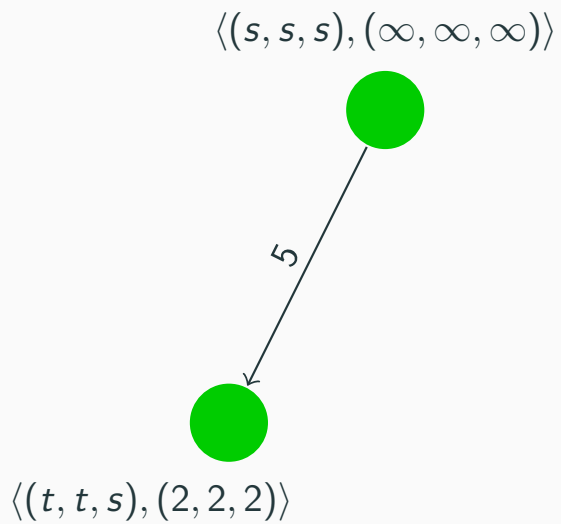

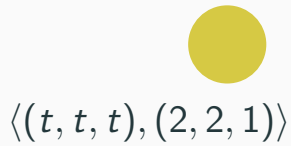

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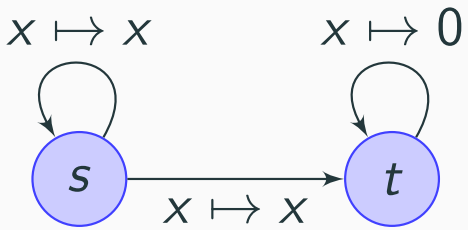

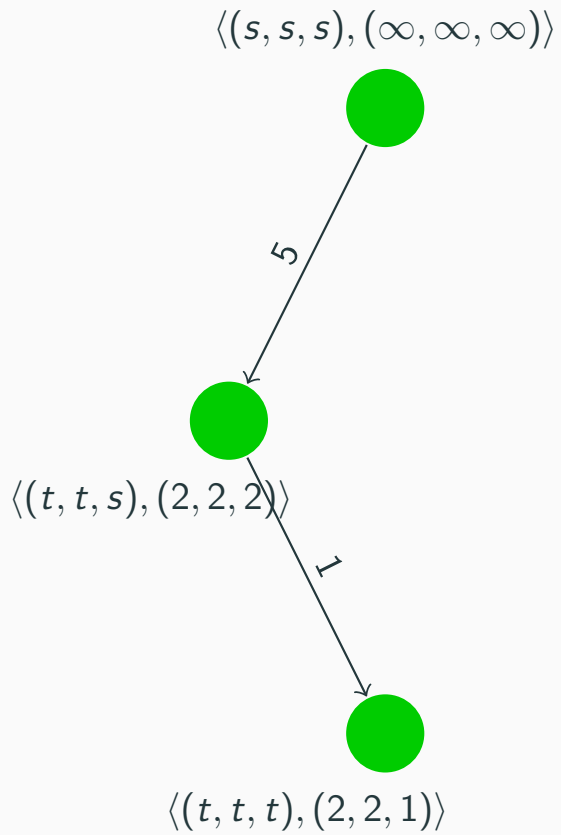

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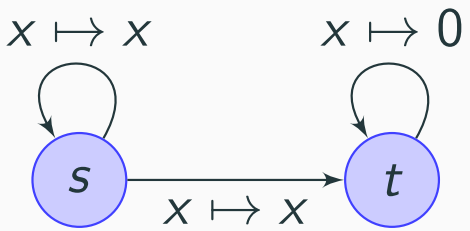

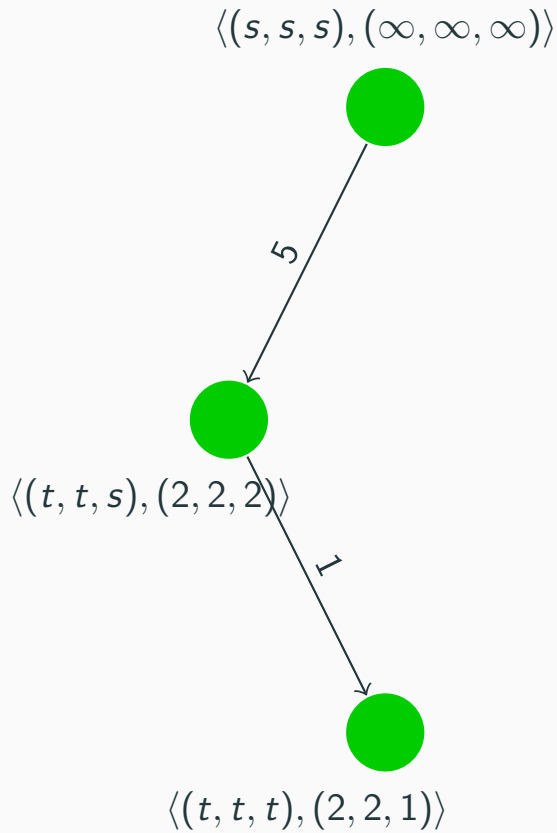

- Along any path, values are non-increasing, and lower bounded by 0 $\implies$ The graph is acyclic.
- For $\exists$ NE, guess such a path, and verify sum of the weights on the edges $\leq M_{\exists}$.
- For $\forall$ NE, change the sign of the weights, guess such a path with social cost $\leq -M_{\forall}$, and verify it. If we get such a path, the answer of $\forall$ NE is "No".
- The values are upper bounded by $|E| \max_{e \in E} cost_e(k)$, at each step we need polynomial space for storing but need EXPTIME to compute $val_{i,c}$, hence decision problem is in EXPSPACE. This implies we can compute PoA, PoS in 2EXPTIME.

Subgame Perfect Equilibria(SPE)

An NE profile $\mathcal{P}$ is an SPE[ab] if for any initialized subgame of the original game, $\mathcal{P}$ works as an NE profile.

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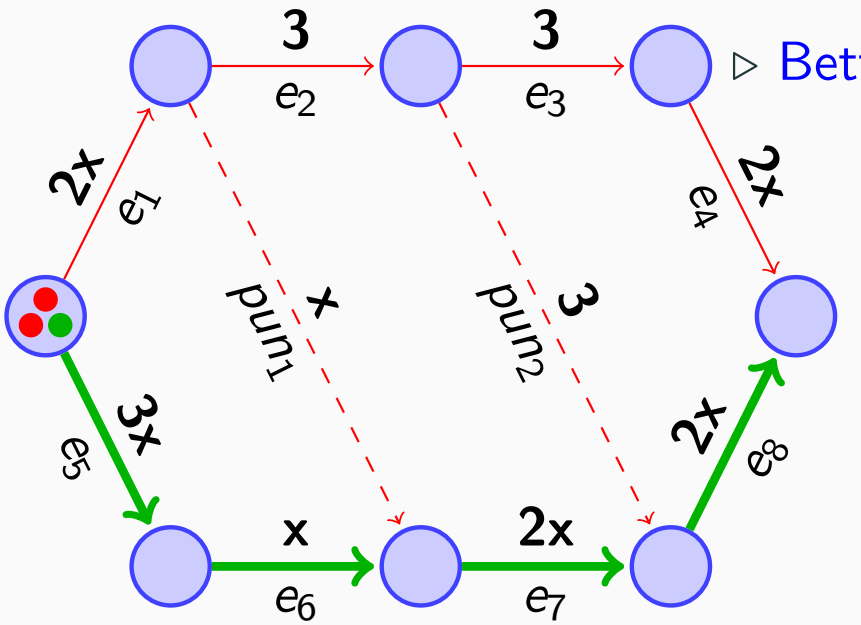

▷ Better Equilibria: Avoids non-credible threats!

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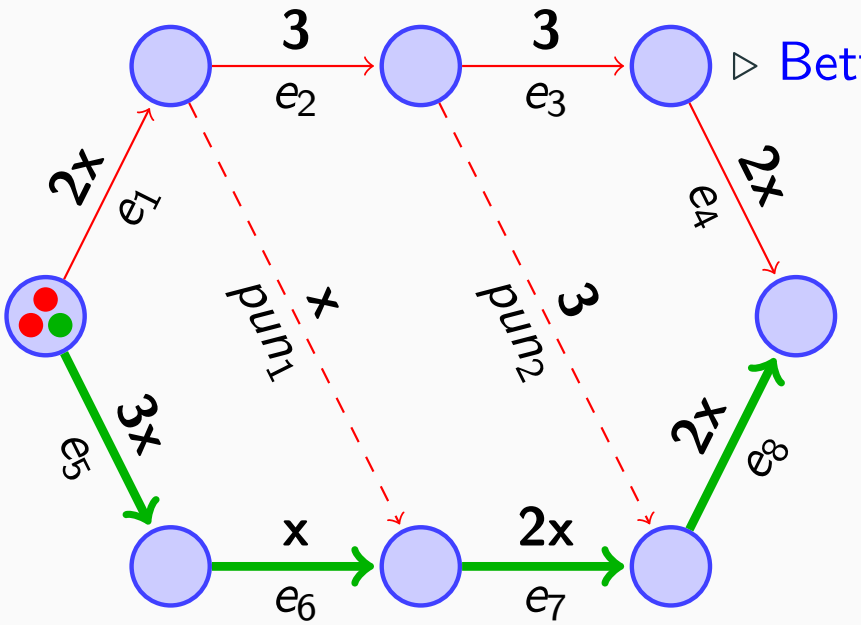

▷ Better Equilibria: Avoids non-credible threats!

▷ Ongoing work: ∃ SPE problem is in EXPSPACE, no decisive conclusion for existence of SPE in the model yet

Conclusion

Recap

- Defined dynamic network games: with Synchronous cost computation.
- Existence of Nash Equilibrium.
- General strategy NE are better than blind NE.
- Computed cost of SO, best and worst NE. $\rightsquigarrow$ 2-EXPTIME upper bound for the decision problem(s) we proposed.
- EXPSPACE algorithm for $\exists$ SPE problem.

Future Objectives

- Finding *better* lower bound complexity result for the decision problem(s).
- We plan to consider number of players as a parameter of the instance, to study whether we can draw any relation between PoS (resp. PoA), NE with this parameter.
- Investigating **Subgame Perfect Equilibria** (SPE) in depth: specifically we are interested whether we can achieve any decisive results on the existence of SPE in our model
- We have plans to consider objectives other than reachability, like regularity.

Thanks!

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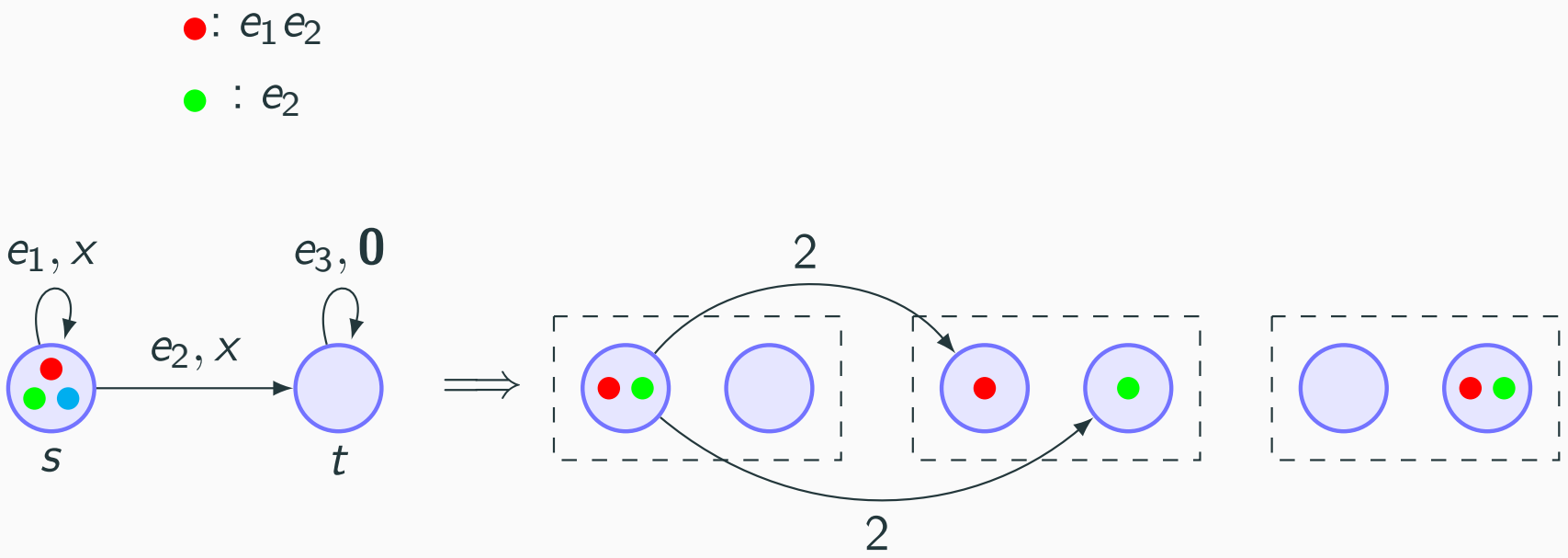
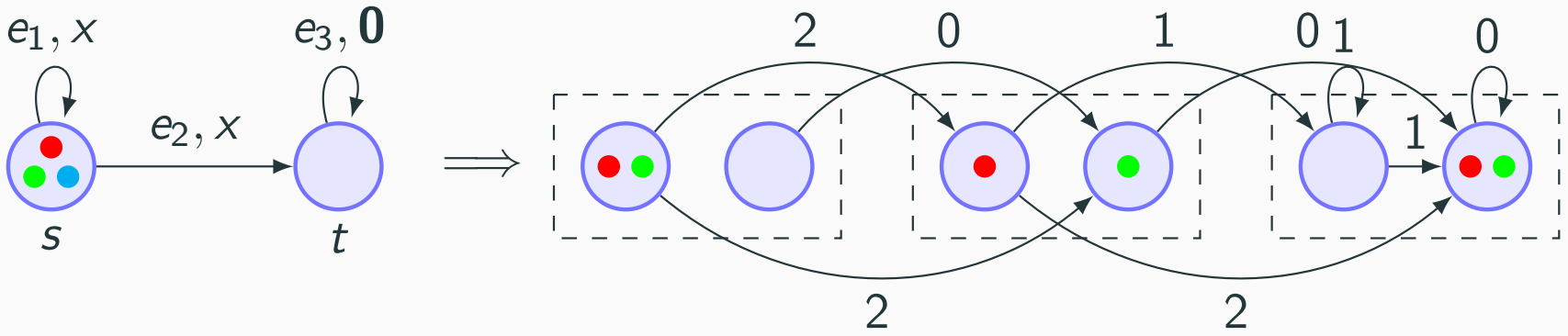


How to solve BR problem?

●: $e_1 e_2$

● : e_2



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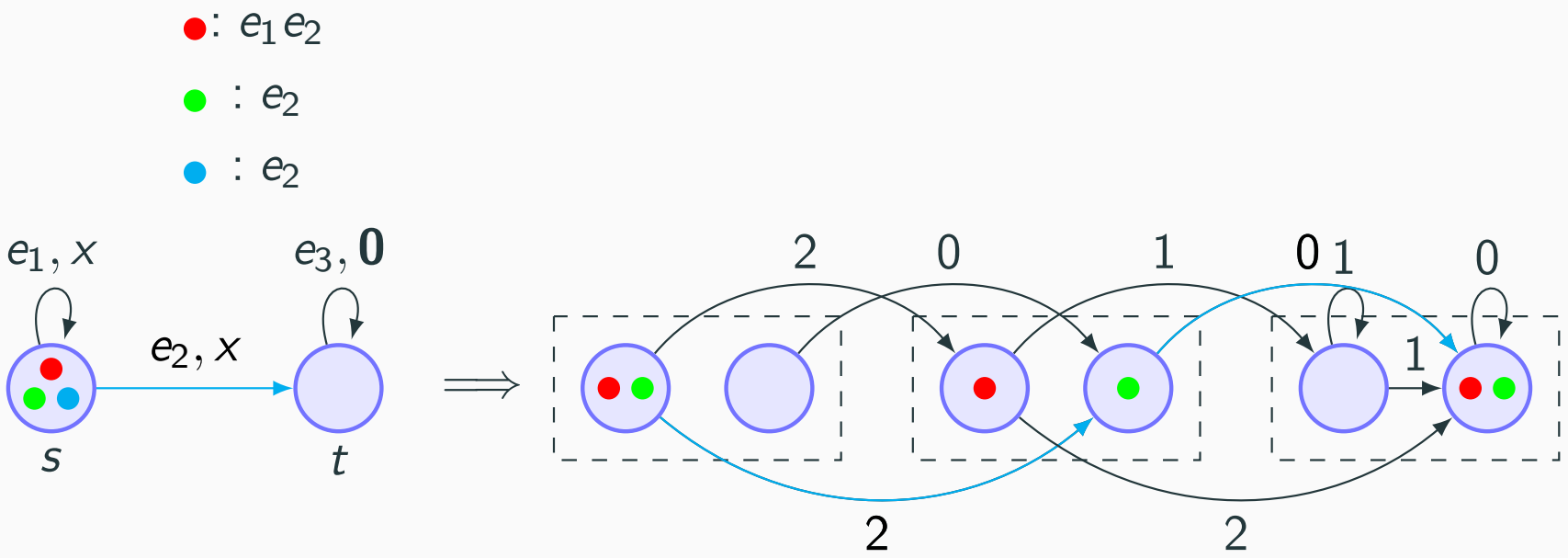

Reduction is polynomial

Weighted shortest path problem for weighted graphs is in PTIME

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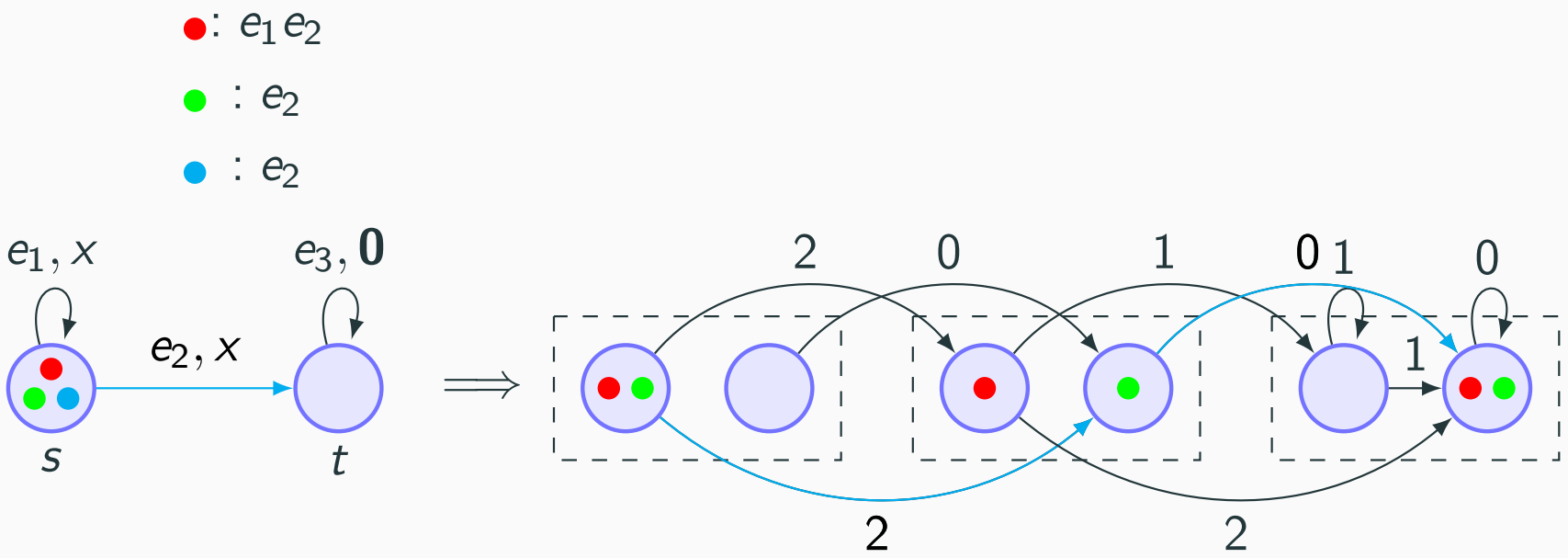

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BR problem for blind strategies is in PTIME

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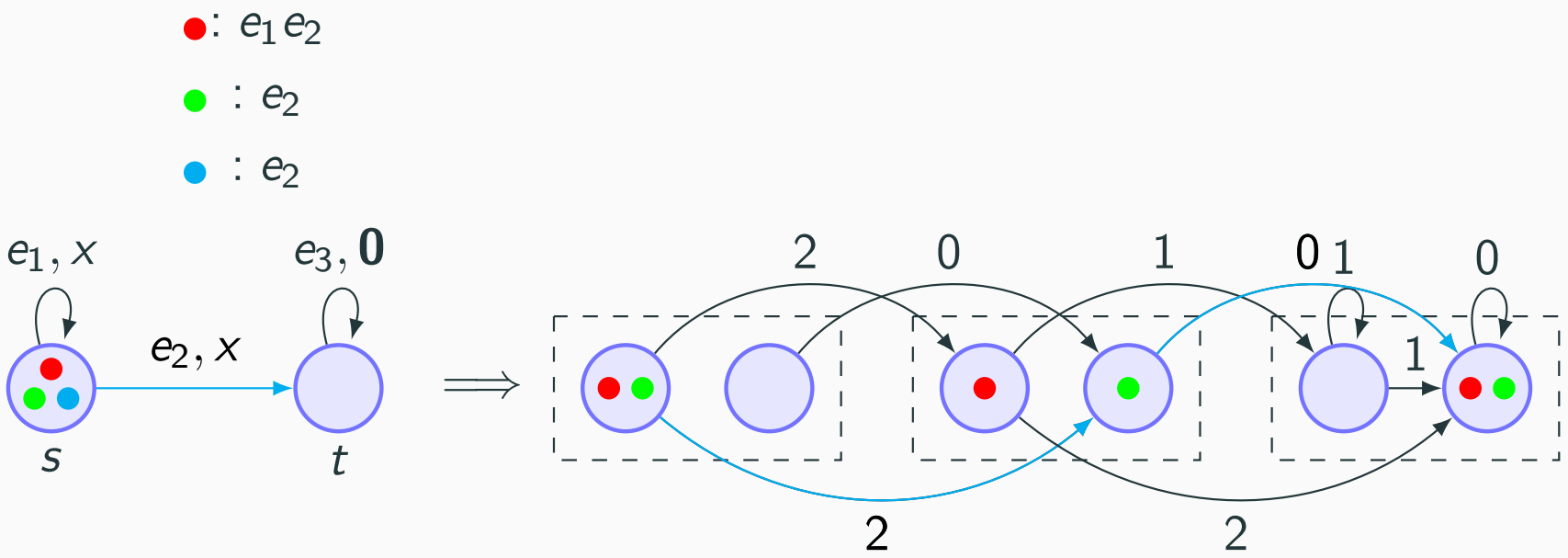

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BR problem for blind strategies is in PTIME

Remark. BR problem for general strategies is also in PTIME!!

Convergence of iterative procedure

Termination is guaranteed by potential function[4]

Potential function $\Pi : \{\text{Set of blind strategy profiles}\} \to \mathbb{N}$

- decreases at each application of BR algorithm
- lower bounded by 0.

For blind strategies

$$\Pi(\mathcal{P}) = \sum_{j=1}^{N_{\mathcal{P}}} \sum_{e \in E} \sum_{i=1}^{load^{j}_{\mathcal{P},e}} cost_e(i)$$

where $N_{\mathcal{P}}$ is the maximum length of strategy "path"
and *load* is the number of players simultaneously using an edge

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Remark. Termination for general strategies is unknown

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