Games Where You Can Play Optimally with Arena-Independent Finite Memory

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Outline

Strategy synthesis for two-player turn-based games

Design optimal controllers for systems interacting with an antagonistic environment.

“Optimal” w.r.t. an objective or a specification.
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“Optimal” w.r.t. an objective or a specification.

Goal: interest in “simple” controllers

Finite-memory determinacy: when do finite-memory controllers suffice?
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Design **optimal** controllers for systems interacting with an antagonistic environment.

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**Goal:** interest in “simple” controllers

Finite-memory determinacy: when do finite-memory controllers suffice?

Inspiration

Results by Gimbert and Zielonka\(^1\) about **memoryless** determinacy.

1. Memoryless determinacy

2. The need for memory

3. Arena-independent finite memory
1 Memoryless determinacy

2 The need for memory

3 Arena-independent finite memory
Two-player turn-based zero-sum games on graphs

- **Finite** two-player arenas: $S_1$ (circles, for $P_1$) and $S_2$ (squares, for $P_2$), edges $E$. 
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- Set $C$ of colors. Edges are colored.
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- “Objectives” given by preference relations $\sqsubseteq \in C^\omega \times C^\omega$ (total preorder). Zero-sum, $\sqsubseteq^{-1}$. 
Two-player turn-based zero-sum games on graphs

- **Finite** two-player arenas: $S_1$ (circles, for $P_1$) and $S_2$ (squares, for $P_2$), edges $E$.
- Set $C$ of colors. Edges are colored.
- “Objectives” given by preference relations $\sqsubseteq \in C^\omega \times C^\omega$ (total preorder). Zero-sum, $\sqsubseteq^{-1}$.
- A strategy for $P_i$ is a (partial) function $\sigma : E^* \rightarrow E$. 
Memoryless determinacy

Question

Given a preference relation, do “simple” strategies suffice to play optimally in all arenas?

A strategy $\sigma$ of $\mathcal{P}_i$ is memoryless if it is a function $\mathcal{Z} S_i \to E$.

![Diagram](image.png)

$C = \{\top, \bot\}$
Memoryless determinacy

**Question**

Given a preference relation, do “simple” strategies suffice to play optimally in all arenas?

A strategy $\sigma$ of $\mathcal{P}_i$ is **memoryless** if it is a function $\mathcal{F} \times S_i \rightarrow E$.

E.g., for reachability, memoryless strategies suffice. Also suffice for safety, Büchi, co-Büchi, parity, mean-payoff, energy, average-energy...
Memoryless determinacy

Good understanding of memoryless determinacy:

- **sufficient** conditions to guarantee memoryless optimal strategies for both players.²,³

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- **sufficient** conditions to guarantee memoryless optimal strategies for both players.\(^2,3\)

- **sufficient** conditions to guarantee memoryless optimal strategies for one player.\(^4,5,6\)

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Memoryless determinacy

Good understanding of memoryless determinacy:

- **sufficient** conditions to guarantee memoryless optimal strategies for both players.\(^2,^3\)

- **sufficient** conditions to guarantee memoryless optimal strategies for one player.\(^4,^5,^6\)

- **characterization** of the preference relations admitting optimal memoryless strategies for both players.\(^7\)

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Gimbert and Zielonka’s characterization\(^8\)

Let \(\sqsubseteq\) be a preference relation. Two results:

1. Characterization of memoryless determinacy w.r.t. properties of \(\sqsubseteq\).

Gimbert and Zielonka’s characterization\textsuperscript{8}

Let $\sqsubseteq$ be a preference relation. Two results:

1. Characterization of memoryless determinacy w.r.t. properties of $\sqsubseteq$.
2. Corollary:

One-to-two-player memoryless lifting

If

- in all one-player arenas of $\mathcal{P}_1$, $\mathcal{P}_1$ has an optimal memoryless strategy,
- in all one-player arenas of $\mathcal{P}_2$, $\mathcal{P}_2$ has an optimal memoryless strategy,

then both players have an optimal memoryless strategy in all two-player arenas.

\textsuperscript{8}Gimbert and Zielonka, “Games Where You Can Play Optimally Without Any Memory”, 2005.
Gimbert and Zielonka’s characterization

Let \( \sqsubseteq \) be a preference relation. Two results:

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Extremely useful in practice. Very easy to recover memoryless determinacy of, e.g., mean-payoff and parity games.

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- **Memoryless determinacy**
- **The need for memory**
- **Arena-independent finite memory**
The need for memory

Memoryless strategies do not always suffice.

• Büchi($A$) $\land$ Büchi($B$): requires finite memory.
The need for memory

Memoryless strategies do not always suffice.

- Büchi($A$) ∧ Büchi($B$): requires finite memory.

\[ A \quad s_1 \quad s_2 \quad B \]

\[ A \quad m_1 \quad m_2 \quad B \]
The need for memory

Memoryless strategies do not always suffice.

\[ \begin{array}{c}
A \\
(1, -1) \\
s_1 \\
\end{array} \quad \begin{array}{c}
(1, -1) \\
(-1, -1) \\
s_2 \\
(-1, 1) \\
B \\
\end{array} \]

- **Büchi**(A) \(\land\) **Büchi**(B): requires **finite memory**.

\[ \begin{array}{c}
A \\
m_1 \\
A \\
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- Mean payoff \(\geq 0\) in both dimensions: requires **infinite memory**.\(^9\)

The need for memory

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\[ (1, -1) \quad (1, -1) \]
\[ (1, 1) \quad (-1, 1) \]
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- Büchi(A) \land Büchi(B): requires **finite memory**.

- Mean payoff \( \geq 0 \) in both dimensions: requires **infinite memory**.\(^9\)

\[ \Rightarrow \textbf{Combinations of objectives} \text{ usually require memory.} \]

An attempt at lifting [GZ05] to FM determinacy

- Lack of a good understanding of finite-memory determinacy.
An attempt at lifting [GZ05] to FM determinacy

• Lack of a good understanding of finite-memory determinacy.

• **Related work**: sufficient properties to preserve FM determinacy in Boolean combinations of objectives.\(^{10}\)

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\(^{10}\)Le Roux, Pauly, and Randour, “Extending Finite-Memory Determinacy by Boolean Combination of Winning Conditions”, 2018.
An attempt at lifting [GZ05] to FM determinacy

- Lack of a good understanding of finite-memory determinacy.

- **Related work**: sufficient properties to preserve FM determinacy in Boolean combinations of objectives.\(^\text{10}\)

- Our approach:

  **Hope**: extend Gimbert and Zielonka’s results

  One-to-two-player lifting for **memoryless** finite-memory determinacy.

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\(^{10}\)Le Roux, Pauly, and Randour, “Extending Finite-Memory Determinacy by Boolean Combination of Winning Conditions”, 2018.
Counterexample

Let $C \subseteq \mathbb{Z}$. $P_1$ wants to achieve a play $\pi = c_1 c_2 \ldots \in C^\omega$ s.t.

$$\limsup_{n} \sum_{i=0}^{n} c_i = +\infty \quad \text{or} \quad \exists \infty n, \sum_{i=0}^{n} c_i = 0.$$  

Optimal FM strategies in one-player arenas...
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... but not in two-player arenas: $P_1$ wins but needs infinite memory.
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Optimal FM strategies in one-player arenas... 
... but not in two-player arenas: $\mathcal{P}_1$ wins but needs infinite memory.

Intuition:
In one-player arenas, $\mathcal{P}_1$ can bound the memory he needs in advance. 
In two-player arenas, $\mathcal{P}_2$ can generate arbitrarily long sequences.
1 Memoryless determinacy

2 The need for memory

3 Arena-independent finite memory
Arena-independent memory

• For Büchi($A$) \land Büchi($B$), this structure suffices to play optimally on all arenas for $\mathcal{P}_1$. 

\[ A \xrightarrow{m_1} B \xrightarrow{m_2} \]

\[ A \xleftarrow{m_1} B \xleftarrow{m_2} \]
Arena-independent memory

- For $\text{Büchi}(A) \land \text{Büchi}(B)$, this structure suffices to play optimally on all arenas for $P_1$.

- The counterexample fails because in one-player arenas, the size of the memory is dependent on the size of the arena.
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• The counterexample fails because in one-player arenas, the size of the memory is dependent on the size of the arena.

• Observation: for many objectives, one fixed memory structure suffices for all arenas.

“For all $A$, does there exist $M$...?”
\rightarrow “Does there exist $M$, for all $A$...?”
Arena-independent memory

• For Büchi(A) \lor Büchi(B), this structure suffices to play optimally on all arenas for \mathcal{P}_1.

\begin{center}
\begin{tikzpicture}
  \node (m1) [diamond, draw] at (0,0) {m_1};
  \node (m2) [diamond, draw] at (1,0) {m_2};
  \draw[->] (m1) edge node {} (m2);
  \draw[->] (m2) edge node {} (m1);
\end{tikzpicture}
\end{center}

• The counterexample fails because in one-player arenas, the size of the memory is dependent on the size of the arena.

• Observation: for many objectives, one fixed memory structure suffices for all arenas.

  “For all \mathcal{A}, does there exist \mathcal{M}...?”
  \rightarrow “Does there exist \mathcal{M}, for all \mathcal{A}...?”

Method: reproducing the approach of Gimbert and Zielonka given a memory structure \mathcal{M}. 
Characterization of arena-independent determinacy

Let $\sqsubseteq$ be preference relation, $\mathcal{M}$ be a memory structure.

1. Characterization of “playing with $\mathcal{M}$ is sufficient” in terms of properties of $\sqsubseteq$. 
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2. Corollary:

One-to-two-player lifting

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In short: the study of one-player arenas is sufficient to determine whether playing with arena-independent finite memory suffices.
Applicability and limits

- **Applies to** objectives with optimal arena-independent strategies:
  - generalized reachability,\(^{11}\)
  - generalized parity,\(^{12}\)
  - window parity,\(^{13}\)
  - lower- and upper-bounded (multi-dimensional) energy games.\(^{14,15}\)

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\(^{11}\)Fijalkow and Horn, “The surprizing complexity of reachability games”, 2010.


Applicability and limits

- **Applies to** objectives with optimal **arena-independent** strategies:
  - generalized reachability,
  - generalized parity,
  - window parity,
  - lower- and upper-bounded (multi-dimensional) energy games.

- **Does not apply to**, e.g., multi-dimension lower-bounded energy objectives: the size of the finite memory depends on the arena.

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Conclusion

Key observation: many objectives require arena-independent memory.
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Contributions

• Characterization of arena-independent finite-memory determinacy.
• One-to-two-player lifting.
• Generalization of Gimbert and Zielonka’s work.
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Future work

Understand (arena-dependent) finite-memory determinacy through the study of one-player arenas.
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Future work

Understand (arena-dependent) finite-memory determinacy through the study of one-player arenas.

Thanks!
Gimbert and Zielonka’s characterization

Let \( \sqsubseteq \) be a preference relation.

Both players admit optimal memoryless strategies in all arenas if and only if

1. \( \sqsubseteq \) and \( \sqsubseteq^{-1} \) are monotone: not sensitive to changing prefixes.

\[
\rho \pi \quad \ldots \quad \rho' \pi
\]

\[
\rho \pi' \quad \ldots \quad \rho' \pi'
\]

2. \( \sqsubseteq \) and \( \sqsubseteq^{-1} \) are selective: mixing cycles is useless.

\[
\rho \pi \quad \ldots \quad \rho\pi
\]

\[
\rho \pi' \quad \ldots \quad \rho\pi'
\]

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Application: memoryless determinacy of mean-payoff

- Colors $C = \mathbb{Z}$. Objective: maximize (for $P_1$) or minimize (for $P_2$) the mean-payoff (average weight by transition).

- In one-player arenas, simply reach and loop around the simple cycle with the highest/lowest mean-payoff (for $P_1/P_2$) $\Rightarrow$ memoryless strategy.

$\implies$ Memoryless strategies also suffice to play optimally in two-player arenas!
Finite memory

Finite memory \(\approx\) memory structure + next-action function.

Memory structure

Memory structure \(\mathcal{M} = (M, m_{\text{init}}, \alpha_{\text{upd}})\): finite set of states \(M\), initial state \(m_{\text{init}}\), update function \(\alpha_{\text{upd}}: M \times C \rightarrow M\).

Example for Büchi(\(A\)) \& Büchi(\(B\)) (not yet a strategy!):

Given an arena \(\mathcal{A} = (S_1, S_2, E)\): next-action function \(\alpha_{\text{nxt}}: S_i \times M \rightarrow E\).
FM example

Playing with memory $\mathcal{M}$ in $\mathcal{A} \approx$ playing memoryless in the arena $\mathcal{A} \times \mathcal{M}$.

$\text{Büchi}(\mathcal{A}) \land \text{Büchi}(\mathcal{B})$:
Characterization of arena-independent finite memory

Let $\sqsubseteq$. Let $M = (M, m_{init}, \alpha_{upd})$.

- We classify prefixes according to $M$:
  for $\rho, \rho' \in C^*$, $\rho \sim_M \rho'$ iff $\alpha_{upd}(m_{init}, \rho) = m = \alpha_{upd}(m_{init}, \rho')$.

- From monotone to $M$-monotone: same with $\rho \sim_M \rho'$.

- Similar extension of selective to $M$-selective by classifying cycles in the memory structure.

**Proposition**

Let $\sqsubseteq$, $M$. $P_1$ and $P_2$ have optimal strategies with memory $M$ in all arenas if and only if $\sqsubseteq$ and $\sqsubseteq^{-1}$ are $M$-monotone and $M$-selective.
Formal definitions of $\mathcal{M}$-monotony and $\mathcal{M}$-selectivity

**Definition ($\mathcal{M}$-monotony)**

Let $\mathcal{M} = (\mathcal{M}, m_{\text{init}}, \alpha_{\text{upd}})$ be a memory skeleton. A preference relation $\sqsubseteq$ is $\mathcal{M}$-monotone if for all $m \in \mathcal{M}$, for all $K_1, K_2 \in \mathcal{R}(C)$,

$$\exists w \in L_{m_{\text{init}}, m}, [wK_1] \sqsubseteq [wK_2] \implies \forall w' \in L_{m_{\text{init}}, m}, [w'K_1] \sqsubseteq [w'K_2].$$

**Definition ($\mathcal{M}$-selectivity)**

Let $\mathcal{M} = (\mathcal{M}, m_{\text{init}}, \alpha_{\text{upd}})$ be a memory skeleton. A preference relation $\sqsubseteq$ is $\mathcal{M}$-selective if for all $w \in C^*$, $m = \widehat{\alpha_{\text{upd}}}(m_{\text{init}}, w)$, for all $K_1, K_2 \in \mathcal{R}(C)$ such that $K_1, K_2 \subseteq L_{m,m}$, for all $K_3 \in \mathcal{R}(C)$,

$$[w(K_1 \cup K_2)^* K_3] \sqsubseteq [wK_1^*] \cup [wK_2^*] \cup [wK_3].$$